Solid Mechanics II

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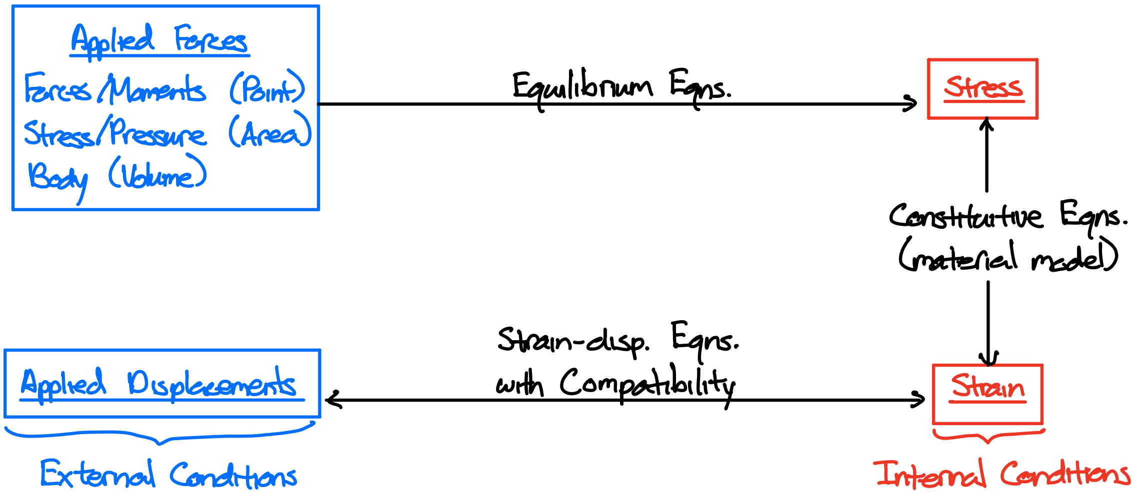
TOPIC 1:
Stress and Equilibrium

In a typical statics course, we sum the forces and moments over a rigid body in order to determine the equilibrium conditions.

Solid mechanics can be considered an extension of statics where we take into account the internal behavior of a deformable material.

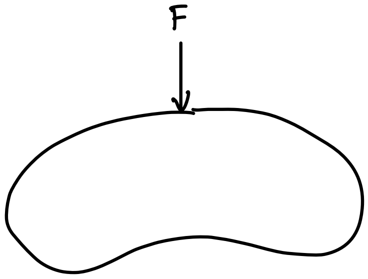
The deformable bodies of solid mechanics are described by PDEs which are easily solved for only very simple cases (e.g., 1D deformation, plane stress [pressure vessels]). For more complicated scenarios, approximate solutions are sought, e.g., the FEM.

Let's review some solid mechanics.

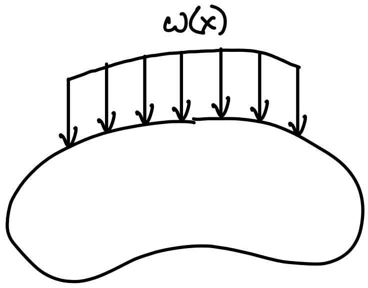


Tonti Diagram

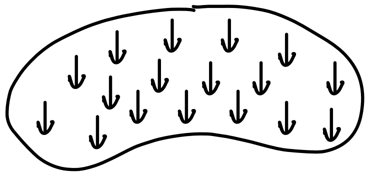
This diagram maps out the solution path for problems in elasticity.



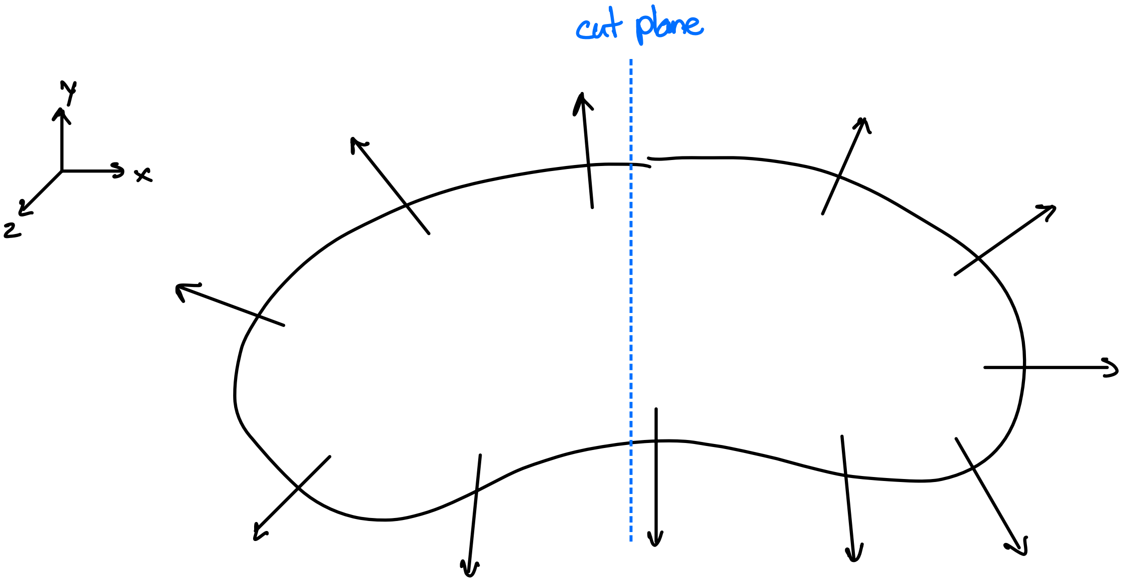
point force

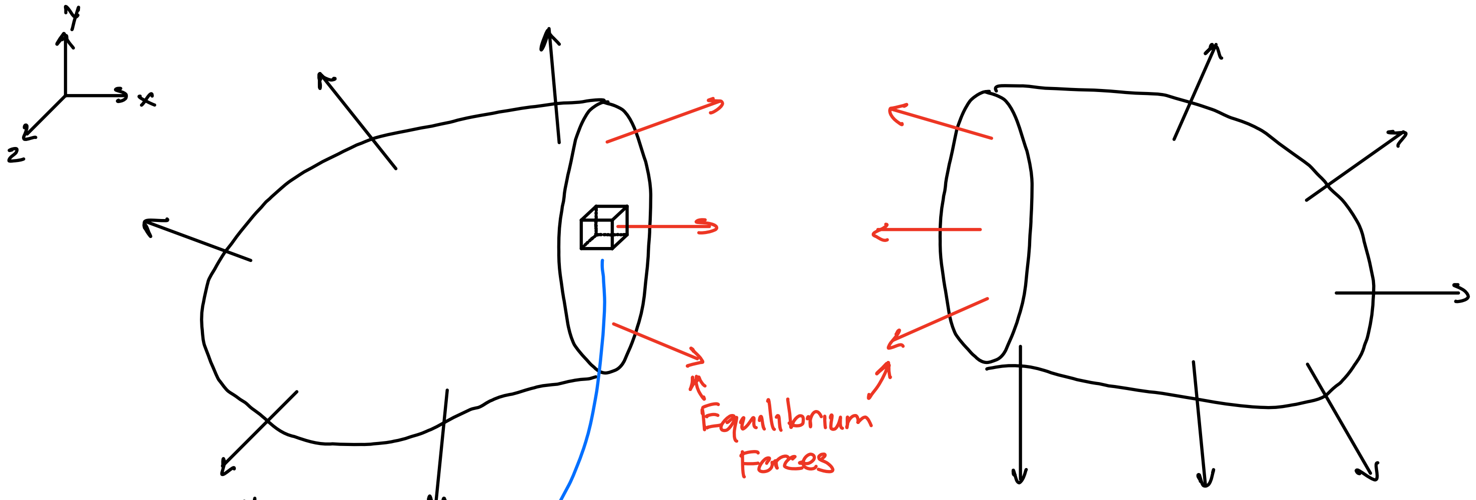


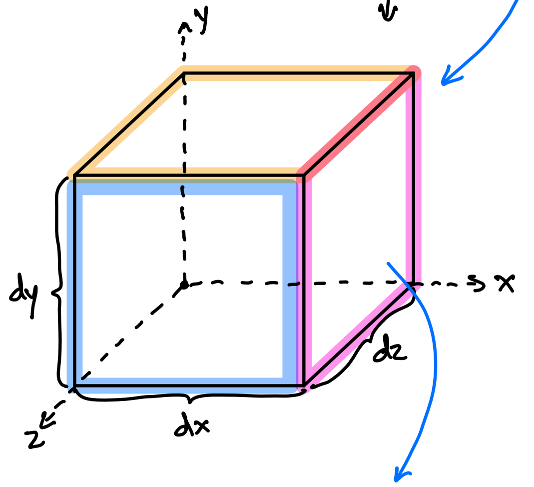
force per unit length/area
(acts over a boundary)



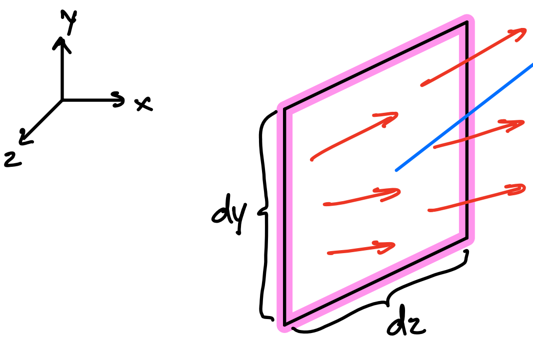
force per unit volume
(acts throughout a body)







Use three perpendicular planes to cut the system through "essentially" the same point, exposing equilibrium forces in the x-, y-, and z-directions.



$\bar{F}_x$: force resultant on yz-plane (x-axis normal)

$\bar{F}$ is a function of area A

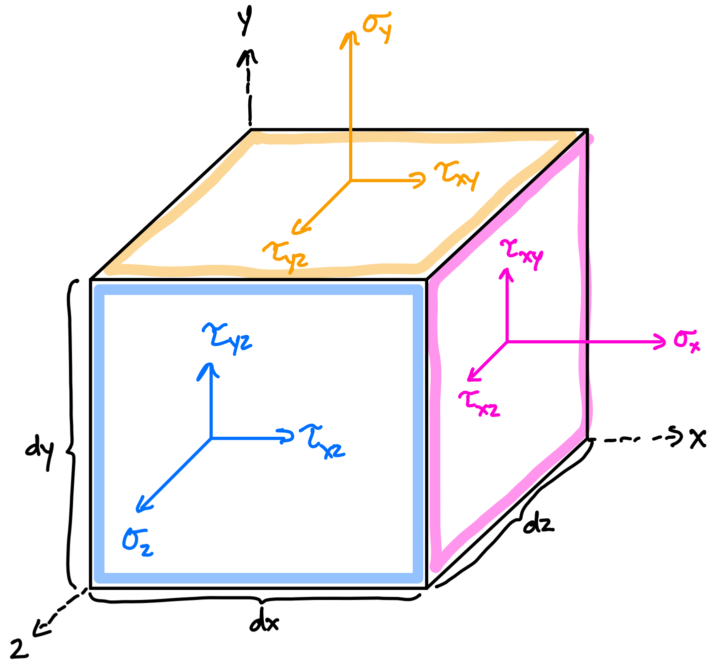
$$\bar{F} = \int \underbrace{\frac{\partial \bar{F}}{\partial A}}_{\bar{T}\text{ : stress traction}} dA$$ total force on area element

$$(\bar{T}_x)_x = \left(\frac{\partial F_x}{\partial A}\right)_x = \sigma_{xx} = \sigma_x \qquad (\bar{T}_x)_y = \left(\frac{\partial F_x}{\partial A}\right)_y = \sigma_{xy} = \tau_{xy} \qquad (\bar{T}_x)_z = \left(\frac{\partial F_x}{\partial A}\right)_z = \sigma_{xz} = \tau_{xz}$$

$$(\bar{T}_y)_x = \left(\frac{\partial F_y}{\partial A}\right)_x = \sigma_{yx} = \tau_{yx} \qquad (\bar{T}_y)_y = \left(\frac{\partial F_y}{\partial A}\right)_y = \sigma_{yy} = \sigma_y \qquad (\bar{T}_y)_z = \left(\frac{\partial F_y}{\partial A}\right)_z = \sigma_{yz} = \tau_{yz}$$

$$(\bar{T}_z)_x = \left(\frac{\partial F_z}{\partial A}\right)_x = \sigma_{zx} = \tau_{zx} \qquad (\bar{T}_z)_y = \left(\frac{\partial F_z}{\partial A}\right)_y = \sigma_{zy} = \tau_{zy} \qquad (\bar{T}_z)_z = \left(\frac{\partial F_z}{\partial A}\right)_z = \sigma_{zz} = \sigma_z$$

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \rightarrow \begin{bmatrix} \bar{T}_x \\ \bar{T}_y \\ \bar{T}_z \end{bmatrix}$$ (stress state)



$$(\bar{F}_x)_x = P_x = \int \overbrace{\left(\frac{\partial F_x}{\partial A}\right)_x}^{\sigma_x} dA$$

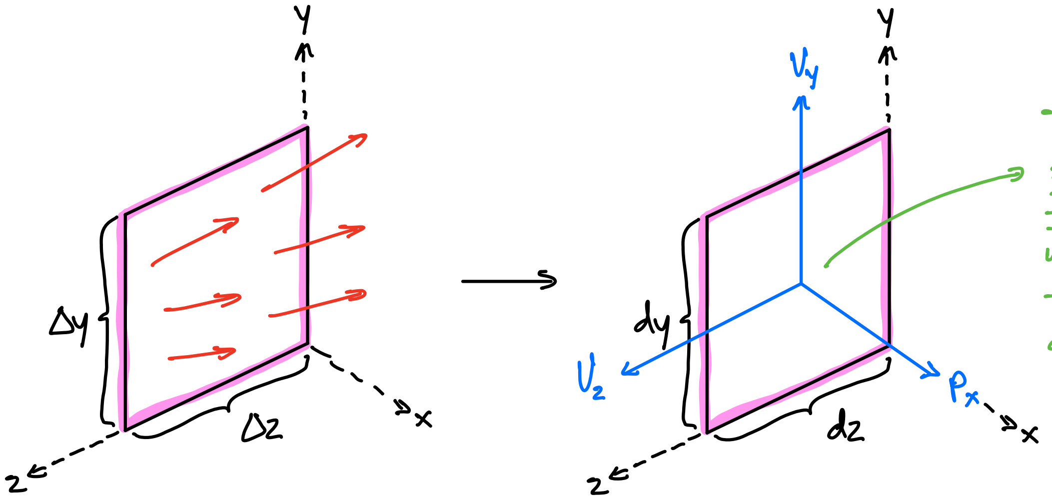
↳ total force normal to yz-plane

$$(\bar{F}_x)_y = V_y = \int \overbrace{\left(\frac{\partial F_x}{\partial A}\right)_y}^{\tau_{xy}} dA$$

↳ total (shear) force parallel to yz-plane and in y-direction

$$(\bar{F}_x)_z = V_z = \int \overbrace{\left(\frac{\partial F_x}{\partial A}\right)_z}^{\tau_{xz}} dA$$

↳ total (shear) force parallel to yz-plane and in z-direction



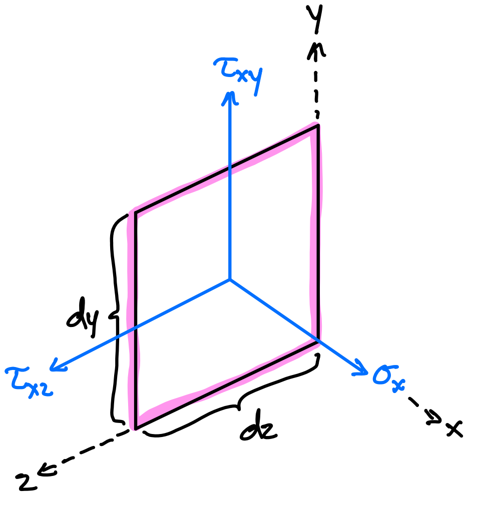
The normal force (P_x) and shear forces (V_y, V_z) act throughout the surface, but we draw them acting through the center just for easy drawing.

$$M_z = -\int (dF_x)_x y = -\int [\sigma_x dA] y = -\int \sigma_x y \, dA$$

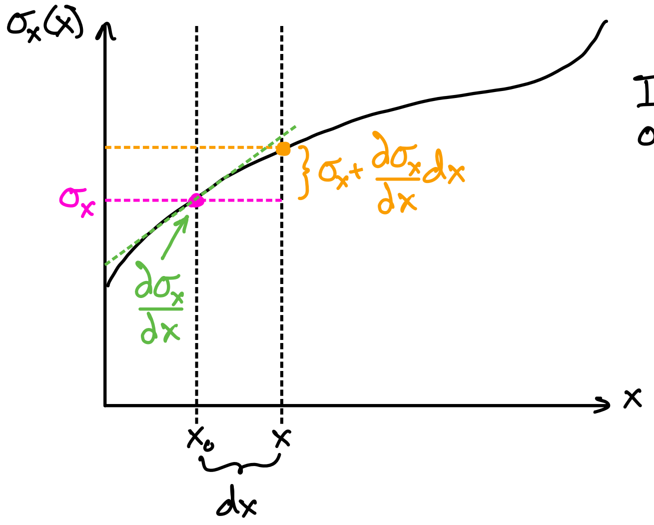
$$M_y = \int (dF_x)_x z = \int [\sigma_x dA] z = \int \sigma_x z \, dA$$

$$M_x = -\int (dF_x)_y z + \int (dF_x)_z y$$

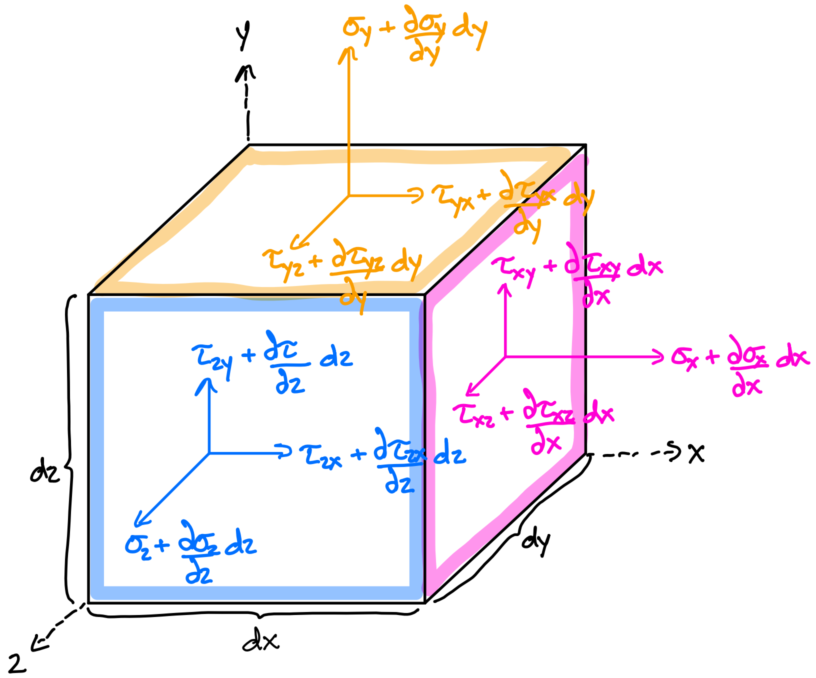
$$= -\int [\tau_{xy} dA] z + \int [\tau_{xz} dA] y = \int (\tau_{xz} y - \tau_{xy} z) dA$$



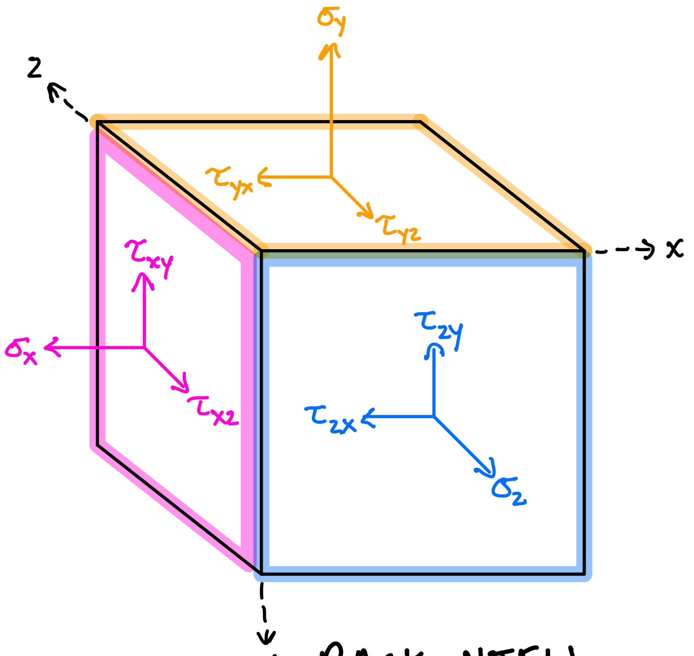
Cutting the system through different points will expose different equilibrium forces; thus the stress is a function of position within the body.



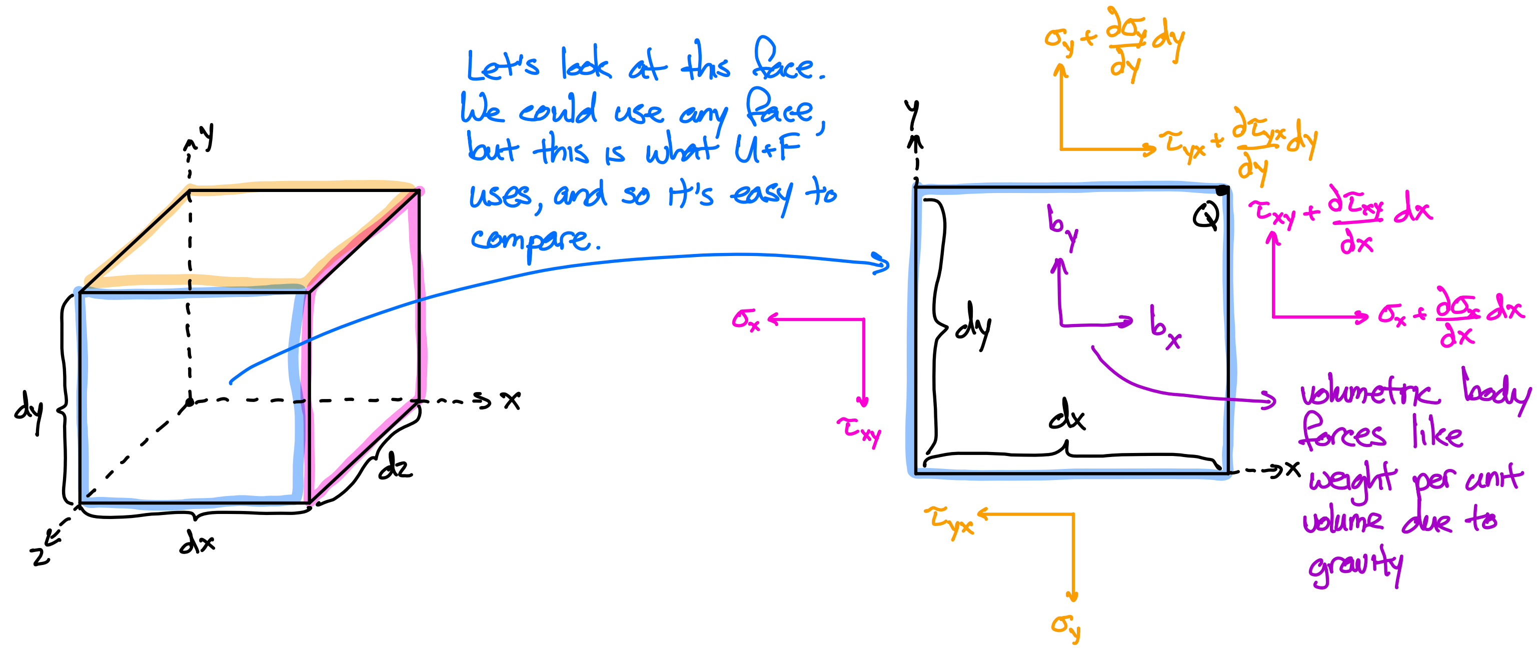
If dx is sufficiently small, then $\sigma_x(x_0) \approx \sigma_x(x_0 + dx) = \sigma_x$; otherwise, $\sigma_x(x_0) = \sigma_x$ and $\sigma_x(x_0 + dx) \approx \sigma_x + \frac{\partial \sigma_x}{\partial x}\Big|_{x=x_0} dx$



FRONT VIEW



BACK VIEW



Let's look at this face. We could use any face, but this is what U+F uses, and so it's easy to compare.

volumetric body forces like weight per unit volume due to gravity

$$\Sigma F_x = -\sigma_x dy dz + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy dz - \tau_{yx} dx dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx dz - \tau_{zx} dx dy + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx dy$$

$$+ b_x dx dy dz = 0$$

$$= \frac{\partial \sigma_x}{\partial x} dx dy dz + \frac{\partial \tau_{yx}}{\partial y} dx dy dz + \frac{\partial \tau_{zx}}{\partial z} dx dy dz + b_x dx dy dz = 0$$

$$\left.\begin{aligned} &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0 \\ \Sigma F_y &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0 \\ \Sigma F_z &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0 \end{aligned}\right\} \text{Equilibrium Eqns.} \qquad \text{(forces only)}$$

$$\Sigma M_Q = -\sigma_x dydz\frac{dy}{2} + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right)dydz\frac{dy}{2} - \tau_{yx}dxdzdy + \tau_{xy}dydzdx + \sigma_y dxdz\frac{dx}{2} - \left(\sigma_y + \frac{\partial \sigma_x}{\partial y}dy\right)dxdz\frac{dx}{2}$$

$$-\tau_{zx}dxdy\frac{dy}{2} + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}dz\right)dxdy\frac{dy}{2} + b_x dxdydz\frac{dy}{2} - b_y dxdydz\frac{dx}{2} = 0$$

$$= \frac{1}{2}\left(\frac{\partial \sigma_x}{\partial x}\cancelto{\approx 0}{dx}\,dy^2dz - \frac{\partial \sigma_y}{\partial y}\cancelto{\approx 0}{dx^2}dydz + \frac{\partial \tau_{zx}}{\partial z}\cancelto{\approx 0}{dxdy^2dz} + b_x\cancelto{\approx 0}{dx}dy^2dz\right) - b_y\cancelto{\approx 0}{dx^2}dydz + (\tau_{xy} - \tau_{yx})dxdzdy = 0$$

$\therefore \tau_{xy} = \tau_{yx}$ similarly $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$

Taking equilibrium into account, we can re-write the stress state:

ignoring equilibrium

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

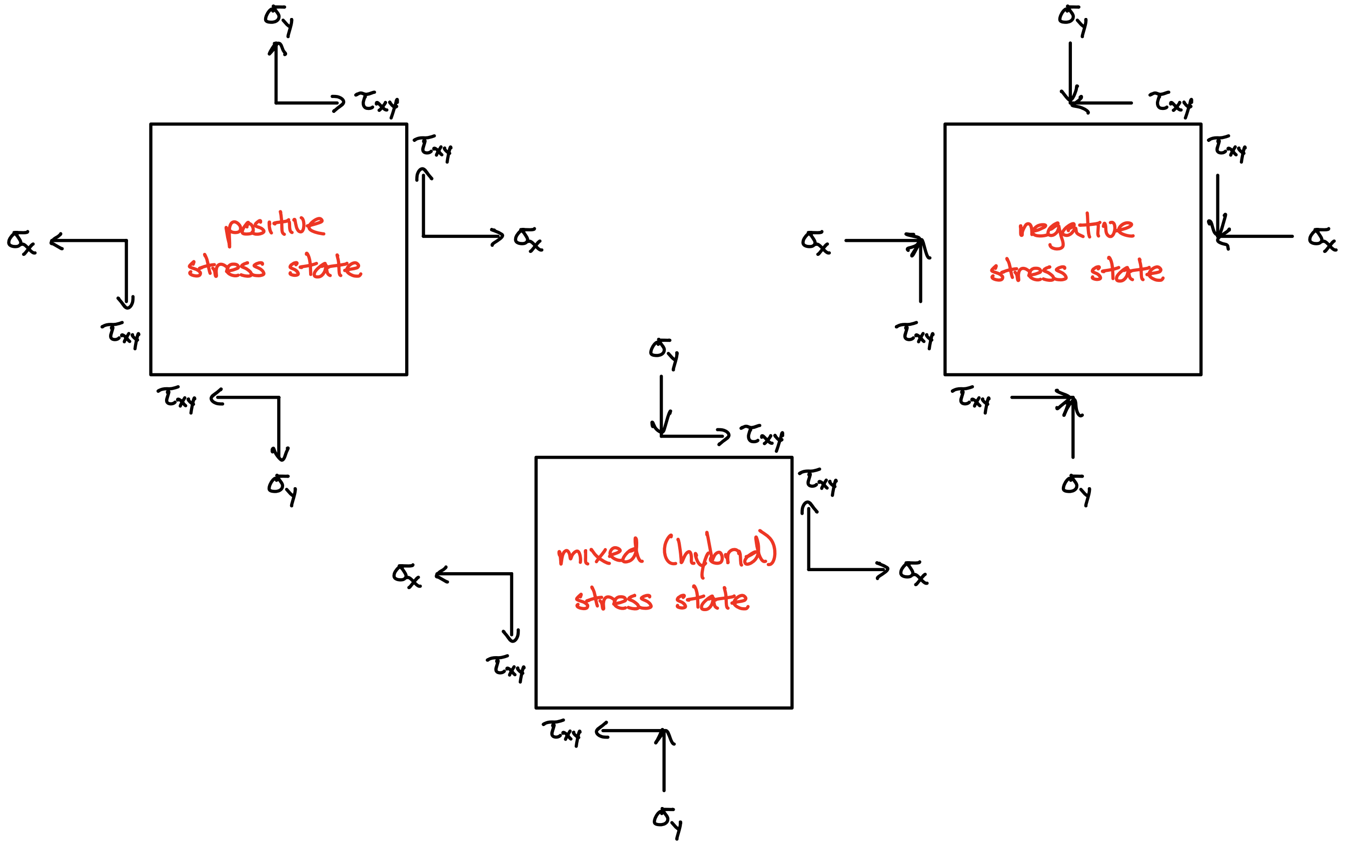
with equilibrium

9 independent terms

6 independent terms
(symmetric matrix)

Sign Convention for Stresses:

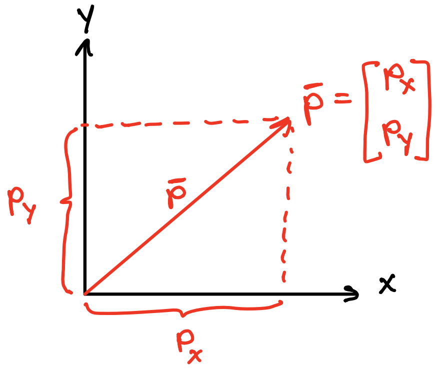
sign of stress = (sign of normal to the plane)·(sign of stress direction)



This also adjusts the sum of forces:

$$\left.\begin{aligned}
\Sigma F_x &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x = 0 \\
\Sigma F_y &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y = 0 \\
\Sigma F_z &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0
\end{aligned}\right\}
\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}}_{D^T}
\underbrace{\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}}_{\bar{\sigma}}
+ \underbrace{\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}}_{\bar{b}}
= D^T \bar{\sigma} + \bar{b} = 0$$

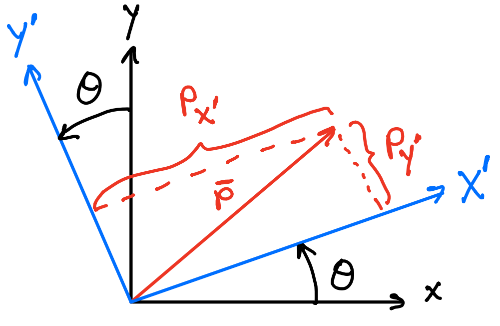
TOPIC 2:
Stress Transformation



$$p_x = |\bar{p}|\cos\theta = \bar{p}\cdot\begin{bmatrix}1\\0\end{bmatrix}$$

$$p_y = |\bar{p}|\cos\left(\frac{\pi}{2}-\theta\right)$$
$$= |\bar{p}|\sin\theta = \bar{p}\cdot\begin{bmatrix}0\\1\end{bmatrix}$$

Nothing special about any particular coordinate system; they are just the framework in which we describe the world. Some coord. sys. are more convenient than others.



$$\hat{n}_{x'}^T = [\cos\theta \quad \sin\theta]$$

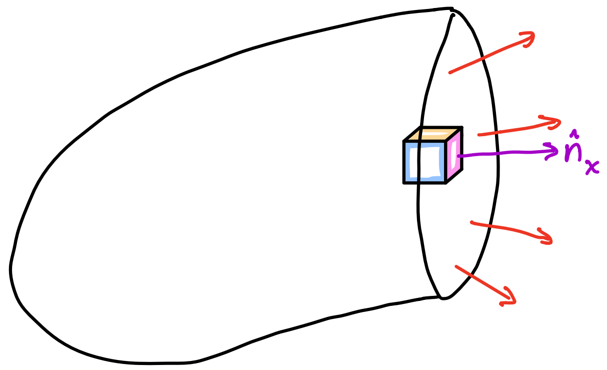
$$\hat{n}_{y'}^T = [\cos(\theta+90^\circ) \quad \sin(\theta+90^\circ)] = [-\sin\theta \quad \cos\theta]$$

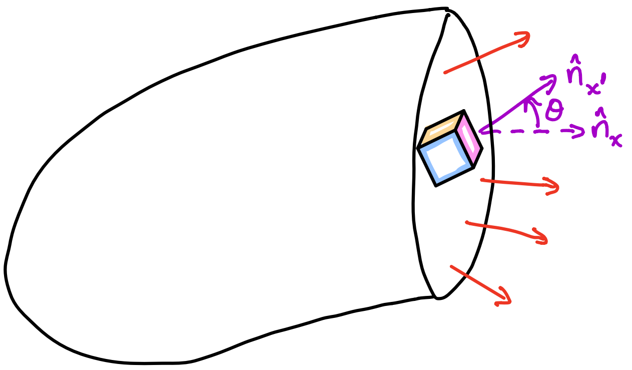
$$\left.\begin{aligned} p_{x'} &= \bar{p}\cdot\hat{n}_{x'} = p_x\cos\theta + p_y\sin\theta \\ p_{y'} &= \bar{p}\cdot n_{y'} = -p_x\sin\theta + p_y\cos\theta \end{aligned}\right\} \begin{matrix}\hat{n}_{x'}^T\\ \hat{n}_{y'}^T\end{matrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\underbrace{\begin{bmatrix}p_x\\ p_y\end{bmatrix}}_{\bar{p}}$$

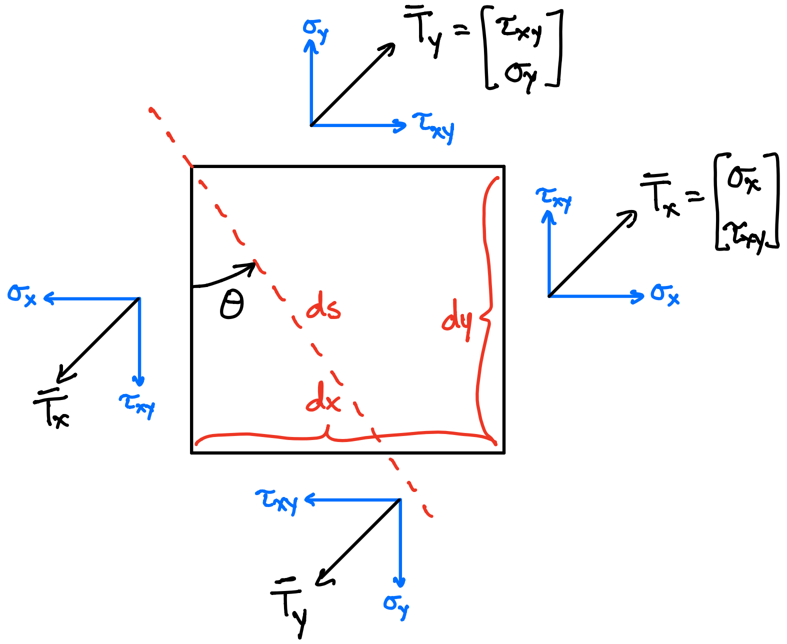
$$|\bar{p}|^2 = p_{x'}^2 + p_{y'}^2 = p_x^2 + p_y^2$$

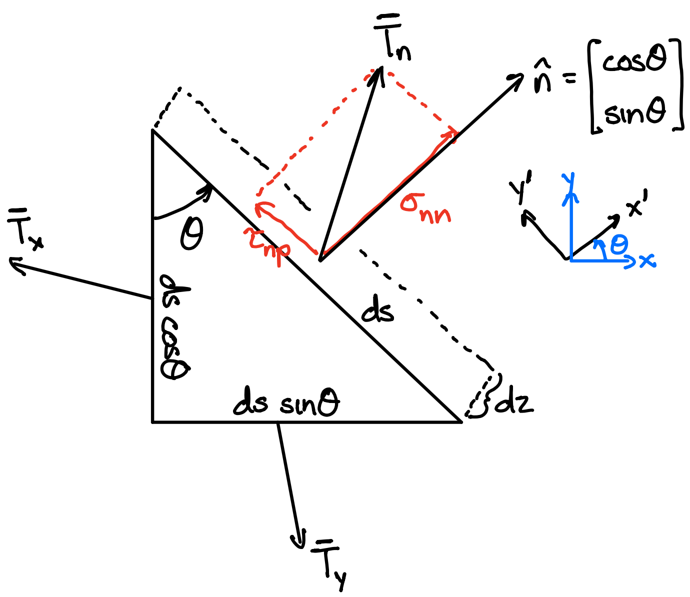
The magnitude of $\bar{p}$ is invariant to rotation

The particular {x,y,z} coord. sys. chosen for the cut planes is not special. An alternative {x',y',z'} sys. will change how the stress state is expressed but each expression is equivalent.









$\bar{T}_n$ balances $\bar{T}_x$ and $\bar{T}_y$ to ensure equilibrium

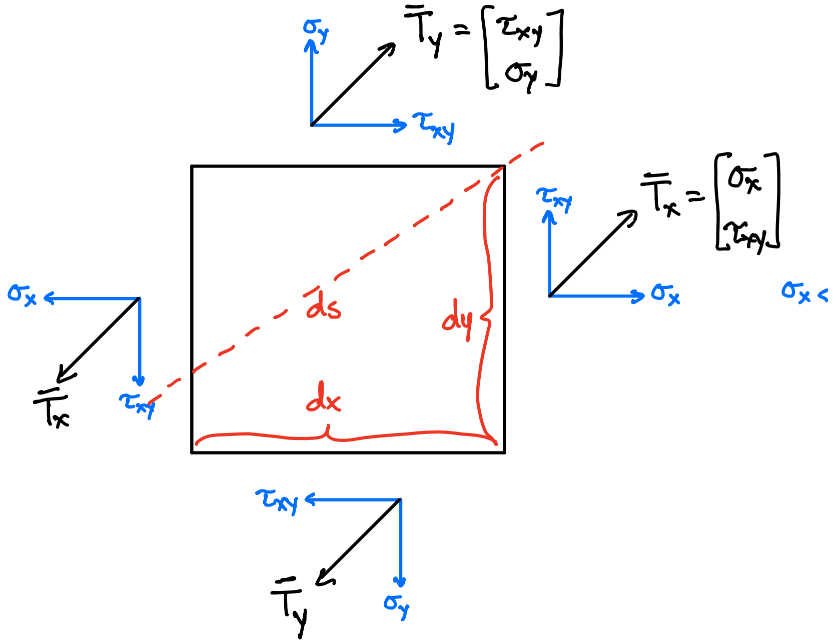
$$\bar{T}_n(ds\,dz) = \bar{T}_x(dz\,ds\cos\theta) + \bar{T}_y(dz\,ds\sin\theta)$$

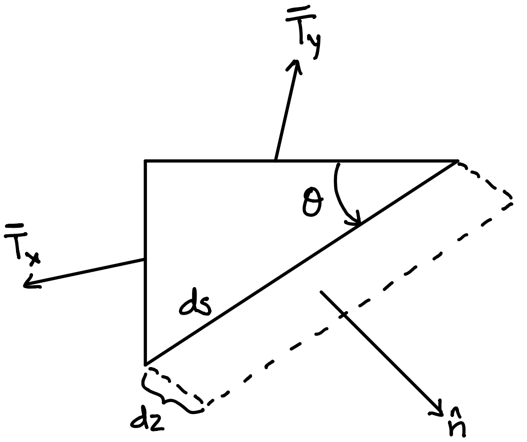
$$\bar{T}_n = \bar{T}_x\cos\theta + \bar{T}_y\sin\theta = \underbrace{\begin{array}{c}\begin{matrix}\bar{T}_x & \bar{T}_y\end{matrix}\\ \begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}\end{array}}_{S}\underbrace{\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}}_{\hat{n}}$$

$$\sigma_{nn} = \bar{T}_n\cdot\hat{n} = \hat{n}^T\bar{T}_n = \hat{n}^T S\hat{n}$$

$$\tau_{np} = \bar{T}_n\cdot\hat{n}_p = \hat{n}_p^T\bar{T}_n = \hat{n}_p^T S\hat{n} \qquad \hat{n}\cdot\hat{n}_p = 0$$

$$\tau_{np}^2 = |T_n|^2 - \sigma_{nn}^2$$





$$\begin{bmatrix}\sigma_{x'}\\ \tau_{x'y'}\end{bmatrix} = \begin{array}{c}\hat{n}^T_{x'}\\ \hat{n}^T_{y'}\end{array}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\overbrace{\underbrace{\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}}_{\hat{n}_{x'}}}^{\bar{T}_{x'}}$$

$$\begin{bmatrix}\tau_{x'y'}\\ \sigma_{y'}\end{bmatrix} = \begin{array}{c}\hat{n}^T_{x'}\\ \hat{n}^T_{y'}\end{array}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\overbrace{\underbrace{\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}}_{\hat{n}_{y'}}}^{\bar{T}_{y'}}$$

$$\underbrace{\begin{bmatrix}\sigma_{x'} & \tau_{x'y'}\\ \tau_{x'y'} & \sigma_{y'}\end{bmatrix}}_{S'} = \underbrace{\begin{array}{c}\hat{n}^T_{x'}\\ \hat{n}^T_{y'}\end{array}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}}_{N^T}\underbrace{\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}}_{S}\underbrace{\begin{array}{c}\begin{matrix}\hat{n}_{x'} & \hat{n}_{y'}\end{matrix}\\ \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\end{array}}_{N}$$

$$\left.\begin{aligned}
\sigma_{x'} &= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\
&= \tfrac{1}{2}(\sigma_x+\sigma_y) + \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \\
\sigma_{y'} &= \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \\
&= \tfrac{1}{2}(\sigma_x+\sigma_y) - \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta \\
\tau_{x'y'} &= (\sigma_y-\sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta-\sin^2\theta) \\
&= -\tfrac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta
\end{aligned}\right\}$$

The stress state in the new coord. sys. is related to the stress state in the old coord. sys.

What value of σ_x ensures a traction-free surface? Assume the traction-free surface has a normal $n^T = [a \;\; b]$.

$$\begin{bmatrix} \sigma_x & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{aligned} \sigma_x a + b &= 0 \\ a + 2b &= 0 \end{aligned} \qquad \therefore\; a = -2b$$

$$\sigma_x(-2b) + b = 0 \qquad \therefore\; \sigma_x = \frac{1}{2}$$

In what direction does it act?

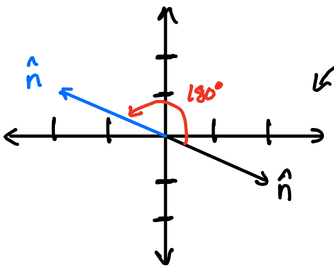
$a = -2b$

$$n = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \therefore\; \underbrace{\hat{n} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

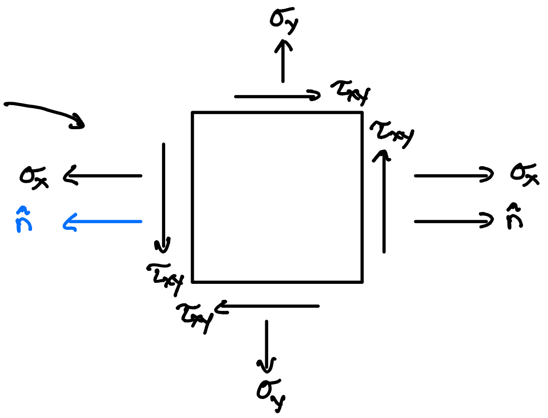
normalization ensures that we do not stretch/shrink the traction vector, $\bar{T}$... that would be unphysical

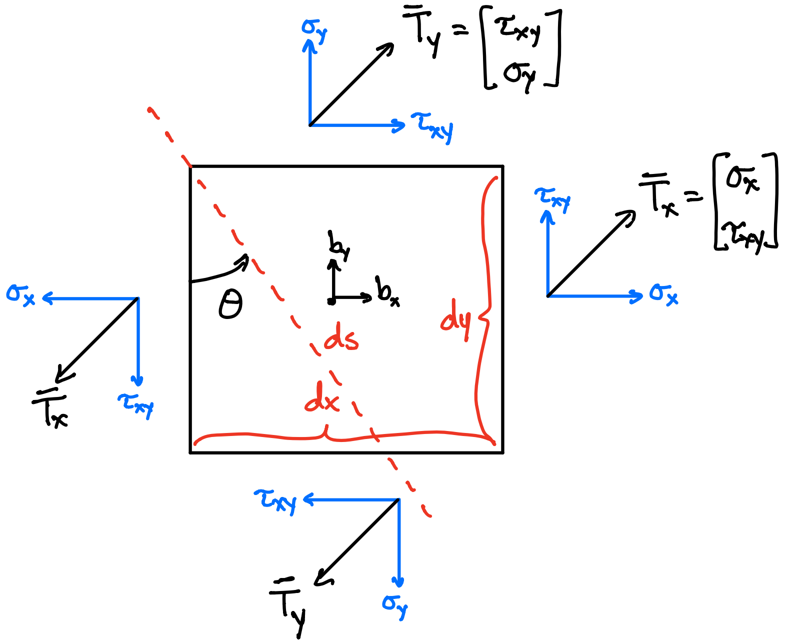
alternatively:

$$n = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \therefore\; \hat{n} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

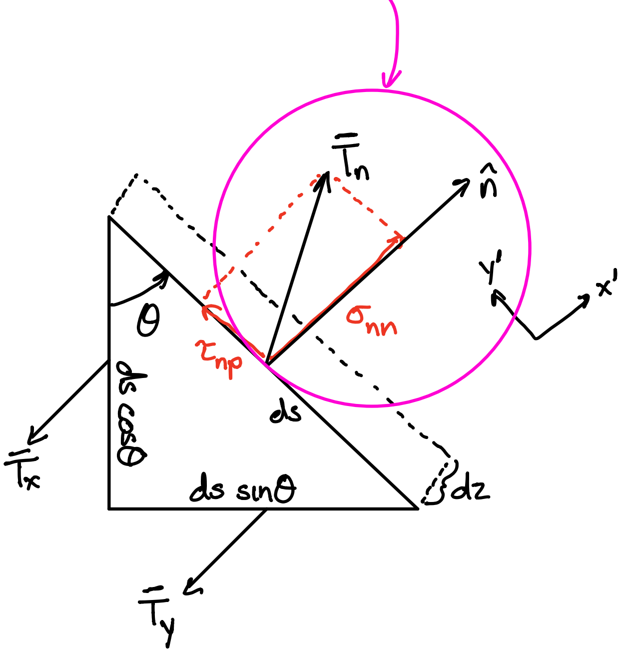


The two vectors are 180° apart. They refer to two opposite, but equivalent planes of the stress element.





$\bar{T}_n = S\hat{n}$ is a vector that, generally does not align with $\hat{n}$. That's why there is a shear stress.



Let's assume that the traction vector, $\bar{T}_n$, points along $\hat{n}$ and has a magnitude σ_p. Then:

$$\bar{T}_n = \sigma_p \hat{n}$$

$S\hat{n} = \sigma_p \hat{n} \quad \therefore \quad (S - I\sigma_p)\hat{n} = 0$ eigenvalue problem asking: in what direction (i.e., orientation of cut plane) does $\bar{T}_n$ align with $\hat{n}$? and what is the magnitude of that traction, σ_p?

$$\begin{bmatrix} \sigma_x - \sigma_p & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma_p \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In 2D:

$$\begin{vmatrix} \sigma_x - \sigma_p & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma_p \end{vmatrix} = \sigma_p^2 - \underbrace{(\sigma_x + \sigma_y)}_{I_1}\sigma_p + \underbrace{\sigma_x\sigma_y - \tau_{xy}^2}_{I_2} = 0$$

stress invariants

In 3D:

$$\begin{vmatrix} \sigma_x - \sigma_p & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_p & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_p \end{vmatrix} = \sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

$I_1 = Tr(S)$; $I_2 = \sum_{i=1}^{3} M_{ii}(S)$; $I_3 = |S|$

Determine the number of unique stress states.

$$\underset{S_1}{\begin{bmatrix}80 & 30\\30 & 40\end{bmatrix}} \quad \underset{S_2}{\begin{bmatrix}96.056 & 0\\0 & 96.056\end{bmatrix}} \quad \underset{S_3}{\begin{bmatrix}60 & -36.056\\-36.056 & 60\end{bmatrix}} \quad \underset{S_4}{\begin{bmatrix}20 & 5\\5 & 100\end{bmatrix}} \quad \underset{S_5}{\begin{bmatrix}40 & -30\\-30 & 80\end{bmatrix}}$$

$S_1 \equiv S_3 \equiv S_5$

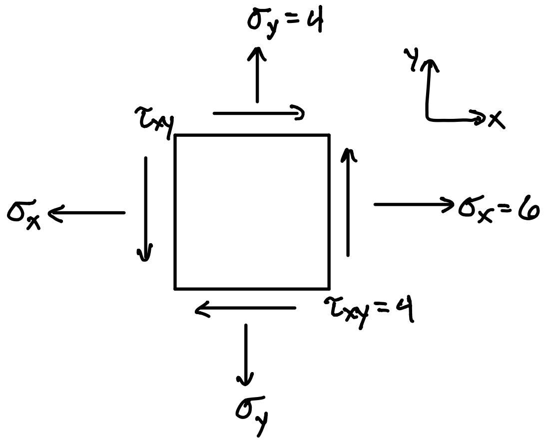
S_2 is unique

S_4 is unique

These all have the same invariants I_1 and I_2, which means that they have the same σ_1 and σ_2, which means that we can draw the same Mohr's circle, which (finally) means that they have all the same $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ for every rotation θ!

Thus there are only two unique stress states since four of them are equivalant.

Determine the principal stresses, σ_1 and σ_2, the principal directions, $\hat{n}_1$ and $\hat{n}_2$, and maximum shear stress τ_{max}.



$$\longrightarrow \quad S = \begin{bmatrix}6 & 4\\4 & 4\end{bmatrix}$$

$$(S - I\sigma_p)\hat{n} = 0$$

$$\begin{vmatrix}6-\sigma_p & 4\\4 & 4-\sigma_p\end{vmatrix} = \sigma_p^2 - 10\sigma_p + 8 = 0 \quad \therefore \quad \sigma_p = \sigma_{1,2} = \frac{10 \pm \sqrt{68}}{2} \qquad \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{68}$$

$$\begin{bmatrix}\sigma_x - \sigma_p & \tau_{xy}\\ \tau_{xy} & \sigma_y - \sigma_p\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = 0 \quad \therefore \quad (\sigma_x - \sigma_p)a + \tau_{xy}b = 0 \quad \therefore \quad \frac{a}{b} = -\frac{\tau_{xy}}{\sigma_x - \sigma_p}$$

Recall that $\hat{n}_p$ is a unit vector, therefore:

$$|\hat{n}_p| = a^2 + b^2 = 1$$

$$a^2 + \left(-\frac{\sigma_x - \sigma_p}{\tau_{xy}}a\right)^2 = 1 \quad \therefore \quad a = \frac{\tau_{xy}}{\sqrt{(\sigma_x - \sigma_p)^2 + \tau_{xy}^2}} \quad \text{and} \quad b = -\frac{\sigma_x - \sigma_p}{\tau_{xy}}a$$

$$\sigma_p = \sigma_1 = \frac{10 + \sqrt{68}}{2} \; ; \quad \hat{n}_p = \hat{n}_1 = \begin{bmatrix}0.788\\0.615\end{bmatrix}$$

$$\sigma_p = \sigma_2 = \frac{10 - \sqrt{68}}{2} \; ; \quad \hat{n}_p = \hat{n}_2 = \begin{bmatrix}0.615\\-0.788\end{bmatrix}$$

Given the stress state, S, below and the normal vector $\hat{n}^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, determine an alternative stress state, S', with x'-axis aligned with $\hat{n}$.

$$S = \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\hat{n}_{y'}$ must be a vector perpendicular to $n_{x'}$; therefore, $\hat{n}_{x'} \cdot \hat{n}_{y'} = 0$.

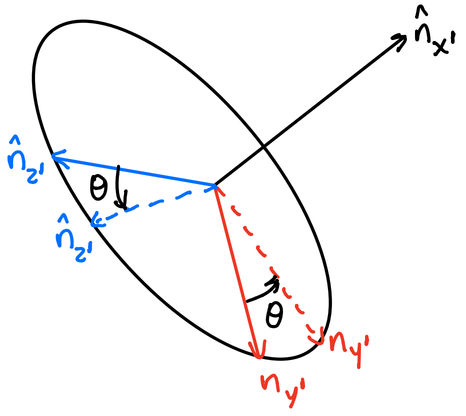
$$\hat{n}^T_{y'} = [a \quad b]$$

$$\hat{n}_{x'} \cdot \hat{n}_{y'} = \frac{\sqrt{2}}{2}a + \frac{\sqrt{2}}{2}b = 0 \qquad \therefore \qquad \frac{a}{b} = -1$$

$$|\hat{n}_{y'}| = a^2 + b^2 = a^2 + (-a)^2 = 2a^2 = 1 \qquad \therefore \qquad a = \frac{\sqrt{2}}{2} \text{ and } b = -\frac{\sqrt{2}}{2}$$

$$\hat{n}^T_{y'} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} \\ \tau_{x'y'} & \sigma_{y'} \end{bmatrix} = N^T S N = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$



$$\sigma_{x'} = \underbrace{\frac{1}{2}(\sigma_x + \sigma_y)}_{\sigma_{avg}} + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \underbrace{\frac{1}{2}(\sigma_x + \sigma_y)}_{\sigma_{avg}} - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Notice that $(\sigma_{x'} - \sigma_{avg})^2 = (\sigma_{y'} - \sigma_{avg})^2$ and so we can treat $\sigma_{x'}$ and $\sigma_{y'}$ generally. We call this general normal stress σ (no subscript).

$$(\sigma_{x'} - \sigma_{avg})^2 = (\sigma_{y'} - \sigma_{avg})^2 = \left[\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2$$

$$\downarrow$$

$$(\sigma - \sigma_{avg})^2 = \left[\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2$$

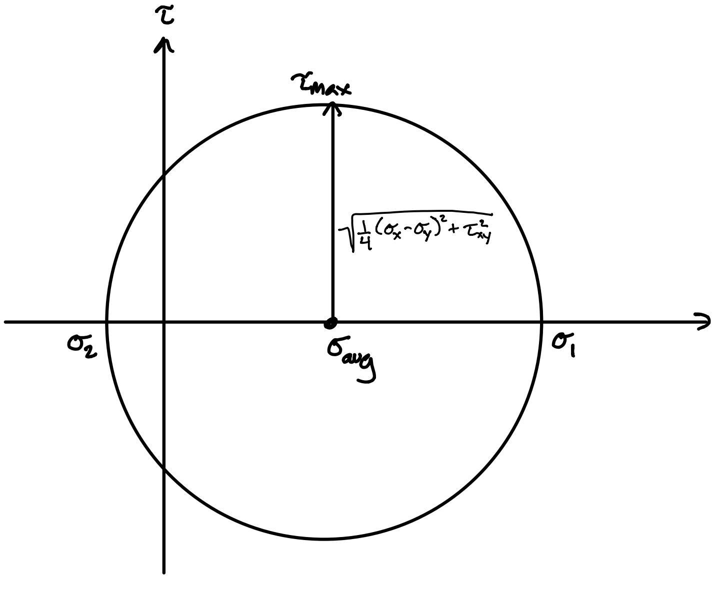
We can drop the subscript from $\tau_{x'y'}$ as well and simply call it τ. If we square it, we get:

$$\tau^2 = \left[-\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2$$

$$(\sigma - \sigma_{avg})^2 + \tau^2 = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2$$

$$(x - c)^2 + y^2 = r^2$$

equation of a circle centered at $\sigma = \sigma_{avg}$ and a radius of $\sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}$.



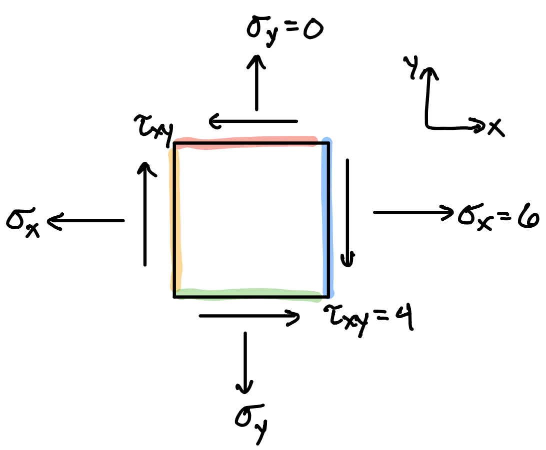
How to draw Mohr's circle:

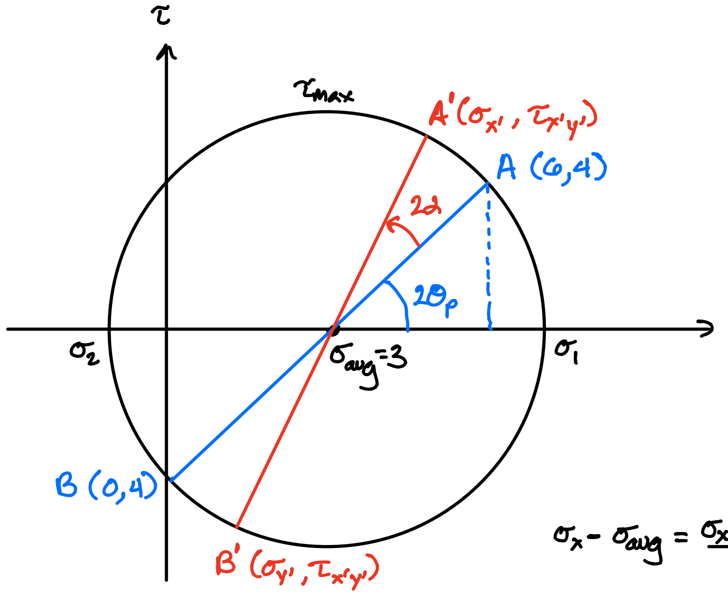
(1) locate $\sigma_{avg} = \frac{1}{2}(\sigma_x + \sigma_y)$ to center the circle

(2) locate point A at $(\sigma, \tau) = (\sigma_x, -\tau_{xy})$; notice, by convention we take the negative of τ_{xy}.

(3) locate point B $(\sigma, \tau) = (\sigma_y, \tau_{xy})$ by drawing a diameter through σ_{avg} and point A.

Using Mohr's circle and the stress state below, determine the stress state following a 10° rotation α.





(1) locate $\sigma_{avg} = \frac{1}{2}(\sigma_x + \sigma_y)$ to center the circle

(2) locate point A at $(\sigma, \tau) = (\sigma_x, -\tau_{xy})$; notice, by convention we take the negative of τ_{xy}.

(3) locate point B $(\sigma, \tau) = (\sigma_y, \tau_{xy})$ by drawing a diameter through σ_{avg} and point A.

$$\sigma_x - \sigma_{avg} = \frac{\sigma_x - \sigma_y}{2}$$

$$\tan(2\theta_p) = \frac{\tau_{xy}}{\sigma_x - \sigma_{avg}} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \therefore \quad \theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = 26.6°$$

$$\tau_{max} = \sqrt{(\sigma_x - \sigma_{avg})^2 + \tau_{xy}^2} = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2} = 5$$

$$\beta = 2\theta_p + 2\alpha = 73.13°$$

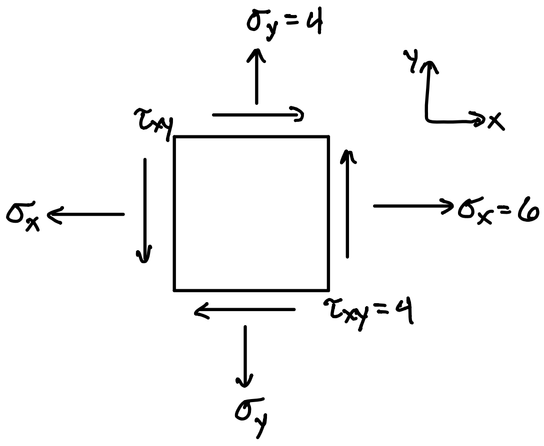
$$\cos\beta = \frac{\sigma_{x'} - \sigma_{avg}}{\tau_{max}} = \frac{\sigma_{avg} - \sigma_{y'}}{\tau_{max}} \quad \therefore \quad \sigma_{x'} = \sigma_{avg} + \tau_{max}\cos\beta = 4.45 \qquad \sigma_{y'} = \sigma_{avg} - \tau_{max}\cos\beta = 1.55$$

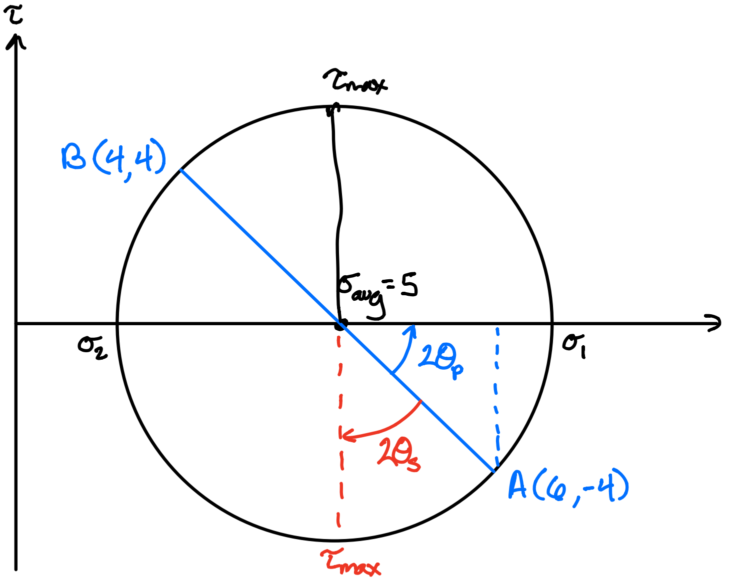
$$\sin\beta = \frac{\tau_{x'y'}}{\tau_{max}} \quad \therefore \quad \tau_{x'y'} = \tau_{max}\sin\beta = 4.78$$

To find the principal stress state, rotate by $-\theta_p$ from the starting stress state.

To obtain the max shear stress state, rotate by $\theta_s = 45° - \theta_p$ from the starting stress state.

Using Mohr's circle and the stress state below, determine the required rotations to reveal the maximum shear stress state, θ_s, and the principal stress state, θ_p. Determine the principal stresses.





(1) locate $\sigma_{avg} = \frac{1}{2}(\sigma_x + \sigma_y)$ to center the circle

$$\sigma_{avg} = \frac{1}{2}(\sigma_x + \sigma_y) = 5$$

(2) locate point A at $(\sigma, \tau) = (\sigma_x, -\tau_{xy})$

(3) locate point B $(\sigma, \tau) = (\sigma_y, \tau_{xy})$

$$\tan(2\theta_p) = \frac{\tau_{xy}}{\sigma_x - \sigma_{avg}}$$

$$\theta_p = \frac{1}{2}\tan\left(\frac{\tau_{xy}}{\sigma_x - \sigma_{avg}}\right) = 37.98^\circ$$

$$\theta_s = 45^\circ - 37.98^\circ = 7.01^\circ$$

To locate the principal stresses, σ_1 and σ_2, we recognize that they are just one radius (i.e., τ_{max}) from σ_{avg}.

$$\tau_{max} = \sqrt{(\sigma_x - \sigma_{avg})^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{17}$$

$$\sigma_{1,2} = \sigma_{avg} \pm \tau_{max} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 5 \pm \sqrt{17}$$

We are often interested in the maximum value of particular stress components as these are often avoided in fear of failure. But the max stress is usually not the one we initially measure in an arbitrary coordinate sys.

The normal stresses ($\sigma_{x'}$ and $\sigma_{y'}$) are maximized when $\tau_{x'y'} = 0$:

$$-\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta = 0 \qquad \therefore \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\theta = \theta_n = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$$ angle of maximum normal stress

$$\sigma_{max,min} = \sigma_{1,2} = \underbrace{\frac{\sigma_x + \sigma_y}{2}}_{\sigma_{avg}} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}\right]^{1/2}$$ principal stresses: by substituting θ_n into $\sigma_{x'x'}$ and $\sigma_{y'y'}$

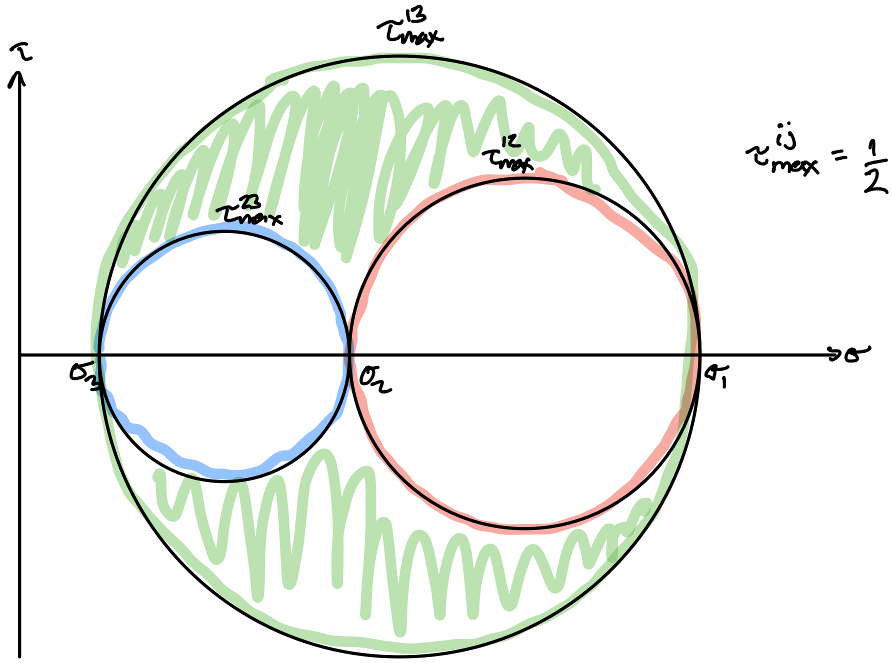
The shear stress is maximized when no component of $\bar{T}$ is normal to the plane (i.e., when $\sigma_{x'}$ and $\sigma_{y'}$ are minimized):

$$\frac{\partial \tau_{x'y'}}{\partial \theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0 \qquad \therefore \tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

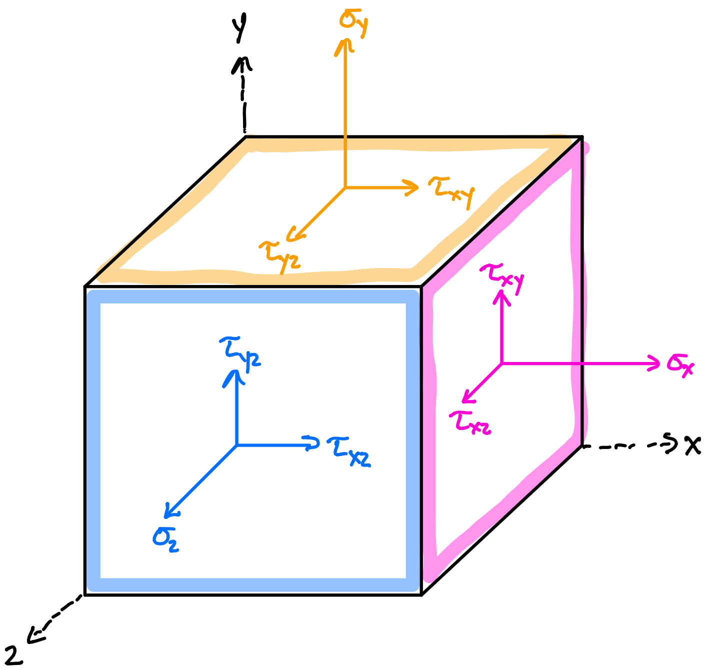
$$\theta = \theta_s = \frac{1}{2}\tan^{-1}\left[-\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right]$$ angle of maximum shear stress

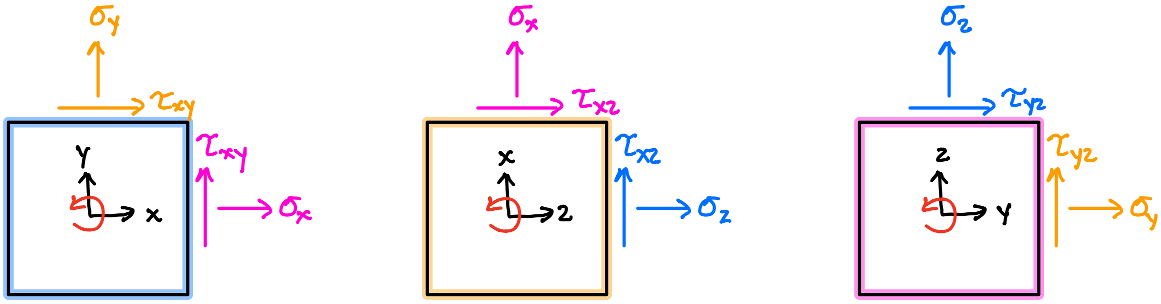
$$\tau_{max} = \pm\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}\right]^{1/2}$$ maximum shear stress: by substituting θ_s into $\tau_{x'y'}$

$$\sigma(\theta_s) = \sigma_{avg}$$

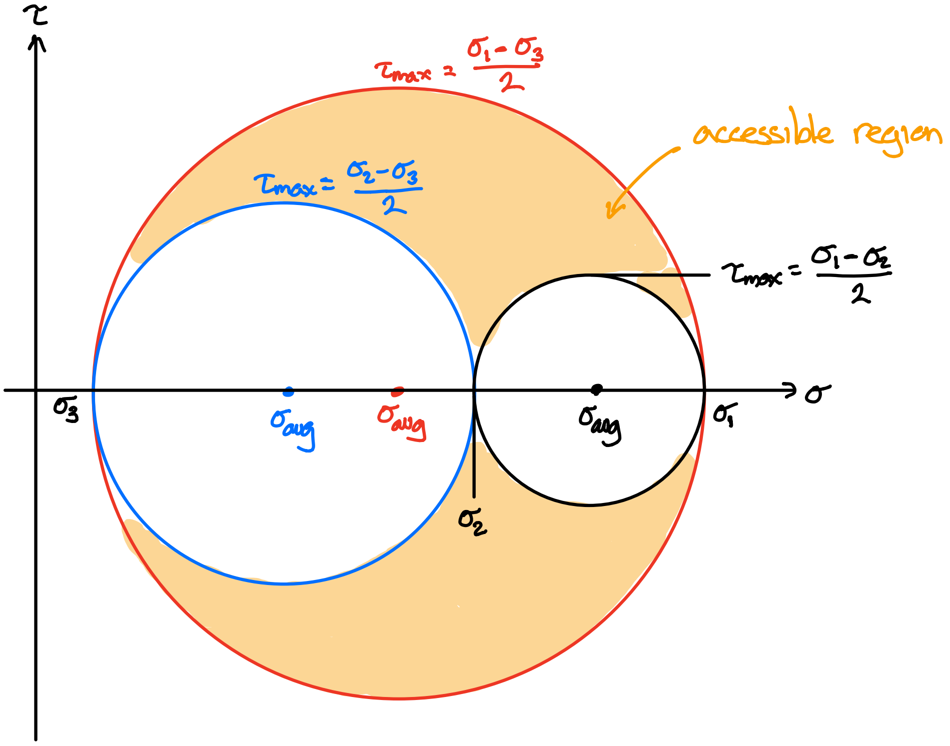


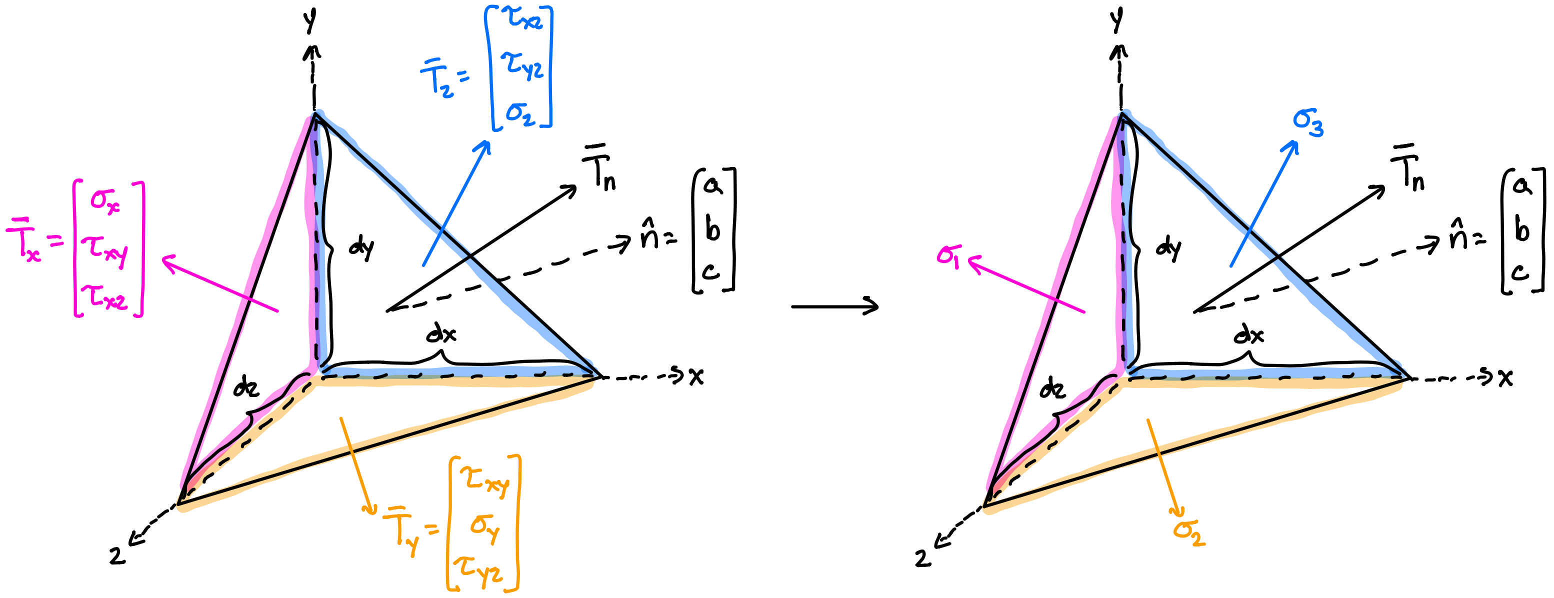
$$\tau_{max}^{ij} = \frac{1}{2}(\sigma_j - \sigma_i)$$





Previously, we discussed Mohr's circle in the context of planar (2D) transformations. The focus was the xy-plane; however, the xz- and yz-plane are also legitamate and generate their own Mohr's circle.





Derivation of Mohr's Circle in 3D:

For simplicity, let a stress state, S, be the principal stress state at point P and let $\hat{n}^T = [a \ b \ c]$ be the normal to a plane through P:

$$S = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \qquad \bar{T}_n = S\hat{n}$$

$$\left.\begin{aligned} \sigma_n &= \hat{n}^T S n = \sigma_1 a^2 + \sigma_2 b^2 + \sigma_3 c^2 \\ \tau_{np}^2 &= |\bar{T}_n|^2 - \sigma_n^2 = \sigma_1^2 a^2 + \sigma_2 b^2 + \sigma_3 c^2 - \sigma_n^2 \\ |\hat{n}| &= a^2 + b^2 + c^2 = 1 \end{aligned}\right\} \text{3 equations, 3 unknowns: } a^2,\ b^2,\ c^2$$

$$\left.\begin{aligned} a^2 &= \frac{N_a}{D_a} = \frac{\tau^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \geq 0 \\[2em] b^2 &= \frac{N_b}{D_b} = \frac{\tau^2 + (\sigma_n - \sigma_3)(\sigma_n - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \geq 0 \\[2em] c^2 &= \frac{N_c}{D_c} = \frac{\tau^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \geq 0 \end{aligned}\right\} \begin{aligned} &\text{Since } \sigma_1 > \sigma_2 > \sigma_3, \text{ then} \\ &D_a > 0,\ D_b < 0, \text{ and } D_c > 0 \end{aligned}$$

$$\left.\begin{aligned}&\tau^2+(\sigma_n-\sigma_2)(\sigma_n-\sigma_3)\geq 0 \qquad \text{add} \quad \tau^2_{max}=\frac{1}{4}(\sigma_2-\sigma_3)^2\\&\tau^2+(\sigma_n-\sigma_2)(\sigma_n-\sigma_3)+\frac{1}{4}(\sigma_2-\sigma_3)^2\geq\frac{1}{4}(\sigma_2-\sigma_3)^2\\&\left[\sigma_n-\frac{1}{2}(\sigma_2+\sigma_3)\right]^2+\tau^2\geq\frac{1}{4}(\sigma_2-\sigma_3)^2\end{aligned}\right\}$$

The inequality defines a region that corresponds to a circle (=) and everything outside (>) that circle.

$$\left.\begin{aligned}&\tau^2+(\sigma_n-\sigma_3)(\sigma_n-\sigma_1)\leq 0 \qquad \text{add} \quad \tau^2_{max}=\frac{1}{4}(\sigma_1-\sigma_3)^2\\&\tau^2+(\sigma_n-\sigma_3)(\sigma_n-\sigma_1)+\frac{1}{4}(\sigma_1-\sigma_3)^2\leq\frac{1}{4}(\sigma_1-\sigma_3)^2\\&\left[\sigma_n-\frac{1}{2}(\sigma_1+\sigma_3)\right]^2+\tau^2\leq\frac{1}{4}(\sigma_1-\sigma_3)^2\end{aligned}\right\}$$

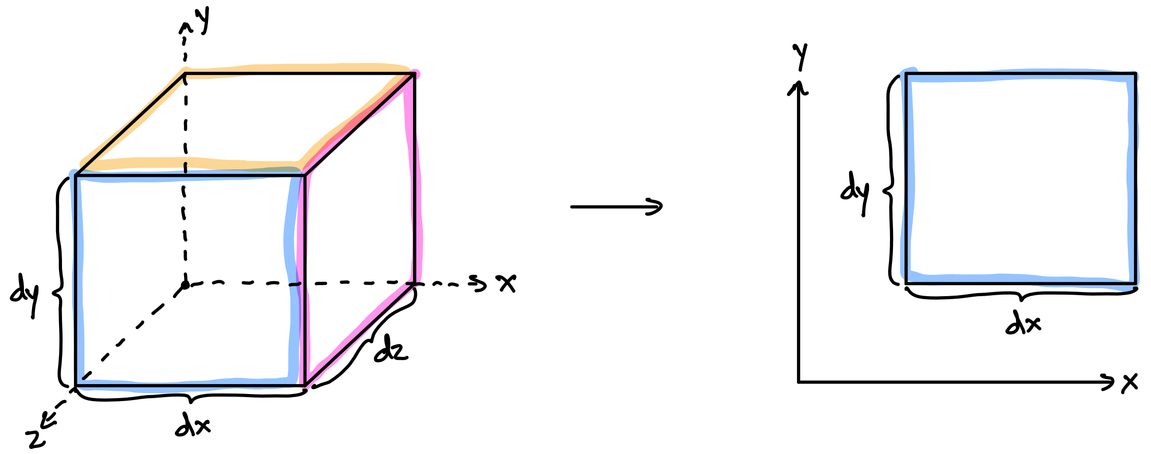
The inequality defines a region that corresponds to a circle (=) and everything inside (<) that circle

$$\left.\begin{aligned}&\tau^2+(\sigma_n-\sigma_1)(\sigma_n-\sigma_2)\geq 0 \qquad \text{add} \quad \tau^2_{max}=\frac{1}{4}(\sigma_1-\sigma_2)^2\\&\tau^2+(\sigma_n-\sigma_1)(\sigma_n-\sigma_2)+\frac{1}{4}(\sigma_1-\sigma_2)^2\geq\frac{1}{4}(\sigma_1-\sigma_2)^2\\&\left[\sigma_n-\frac{1}{2}(\sigma_1+\sigma_2)\right]^2+\tau^2\geq\frac{1}{4}(\sigma_1-\sigma_2)^2\end{aligned}\right\}$$

The inequality defines a region that corresponds to a circle (=) and everything outside (>) that circle.

TOPIC 3:
Displacement and Strain

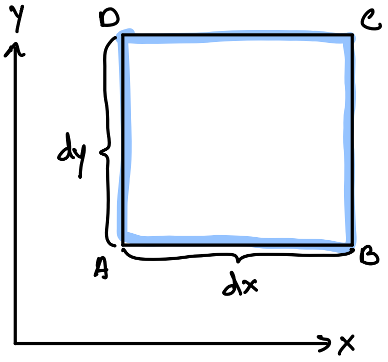
In our study of stress, we never considered the response of the body (e.g., potato, material cube) due to the applied loading. We expect the body to deform, i.e., change size and shape which involves material displacement.



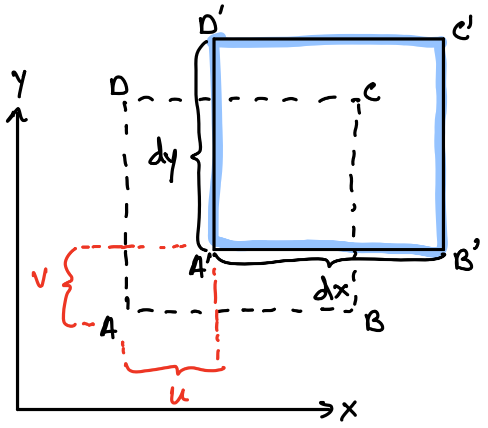
rigid body motion: every bit of material displaces the same amount. The relative displacement is zero.

deformation: bits of material (depending on position) displace a different amount. The relative displacement is non-zero.

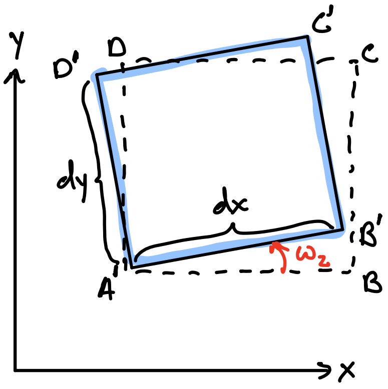
Initial Configuration



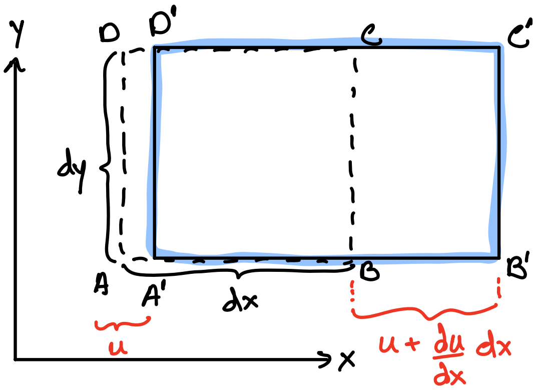
Rigid Body Displacement



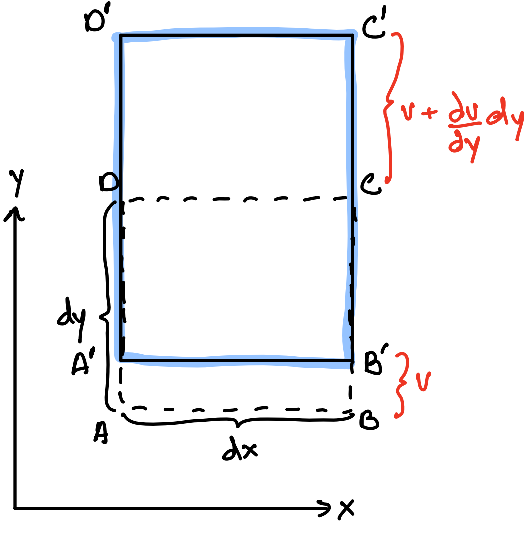
Rigid Body Rotation



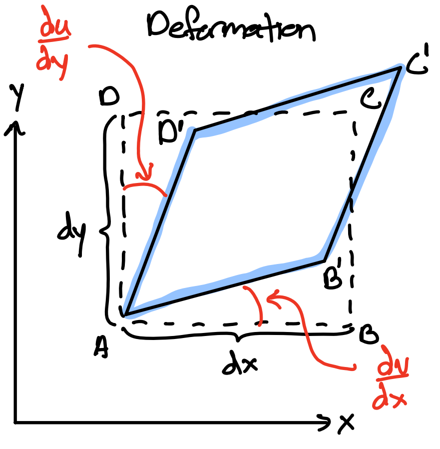
Deformation



Deformation



Deformation



normal strain: deformation along (i.e., parallel to) an axis.

$$\varepsilon_x = \frac{u_B - u_A}{dx} = \frac{u + \frac{\partial u}{\partial x}dx - u}{dx} = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{v_D - v_A}{dy} = \frac{v + \frac{\partial v}{\partial y}dy - v}{dy} = \frac{\partial v}{\partial y}$$

shear strain: deformation that changes the angle between two originally perpendicular axes.

$\tan\alpha = \dfrac{\frac{\partial v}{\partial x}dx}{dx} = \dfrac{\partial v}{\partial x}$ by small angle assumption, $\alpha \approx \dfrac{\partial v}{\partial x}$

$\tan\beta = \dfrac{\frac{\partial u}{\partial y}dy}{dy} = \dfrac{\partial u}{\partial y}$ by small angle assumption, $\beta \approx \dfrac{\partial u}{\partial y}$

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} \quad \frac{\partial w}{\partial y}$$

Strain is not only elongation/contraction ($\varepsilon_x, \varepsilon_y, \varepsilon_z$) but rotation as well ($\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$).

Stress at a point is determined by 3 perpendicular planes and so is strain.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad u_1 = u,\ u_2 = v,\ u_3 = w \quad x_1 = x,\ x_2 = y,\ x_3 = z$$

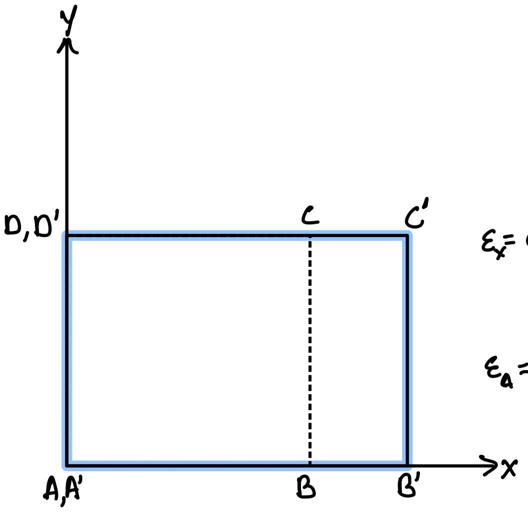
$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x} \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\gamma_{xy} = \varepsilon_{21}$$

γ_{xy} : engineering shear strain

$\frac{1}{2}\gamma_{xy}$: mathematical shear strain

$$\varepsilon_{ij} = \varepsilon = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}$$

normal strain: deformation along (i.e., parallel to) an axis.

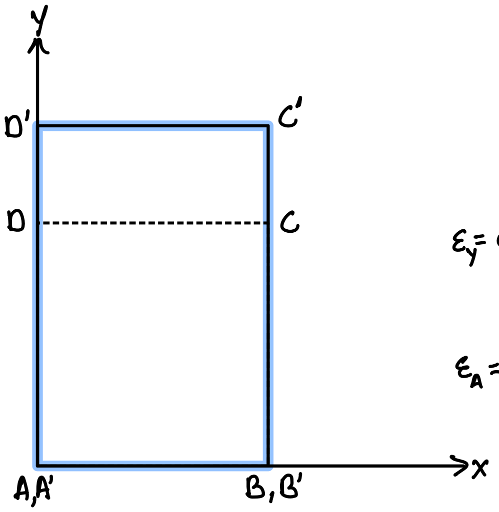


follows the denominator order

$$\varepsilon_x = \frac{du}{dx} \approx \frac{\Delta u}{\boxed{\Delta x}} = \frac{\boxed{u_2 - u_1}}{x_2 - x_1}$$

must be positive

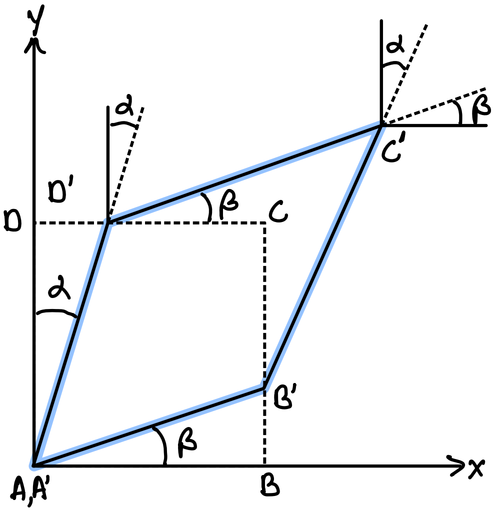
$$\varepsilon_A = \varepsilon_B = \frac{u_B - u_A}{x_B - x_A}$$



$$\varepsilon_y = \frac{dv}{dy} \approx \frac{\Delta v}{\Delta y} = \frac{v_2 - v_1}{y_2 - y_1}$$

$$\varepsilon_A = \varepsilon_D = \frac{v_D - v_A}{y_D - y_A}$$

shear strain: deformation that changes the angle between two originally perpendicular axes.



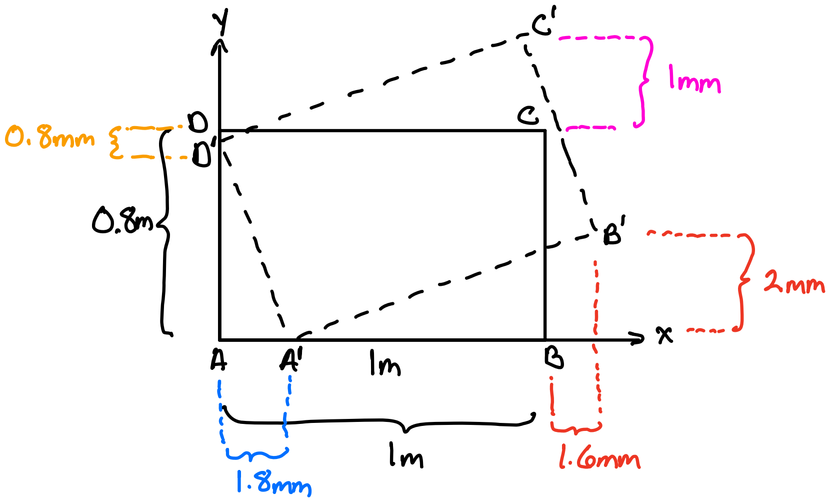
$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} \approx \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x} = \frac{u_2 - u_1}{y_2 - y_1} + \frac{v_2 - v_1}{x_2 - x_1}$$

$$\gamma_{xy}^{A} = \frac{u_D - u_A}{y_D - y_A} + \frac{v_B - v_A}{x_B - x_A}$$

$$\gamma_{xy}^{C} = \frac{u_C - u_B}{y_C - y_B} + \frac{v_C - v_D}{x_C - x_D}$$

$$\gamma_{xy}^{D} = \frac{u_D - u_A}{y_D - y_A} + \frac{v_C - v_D}{x_C - x_D}$$

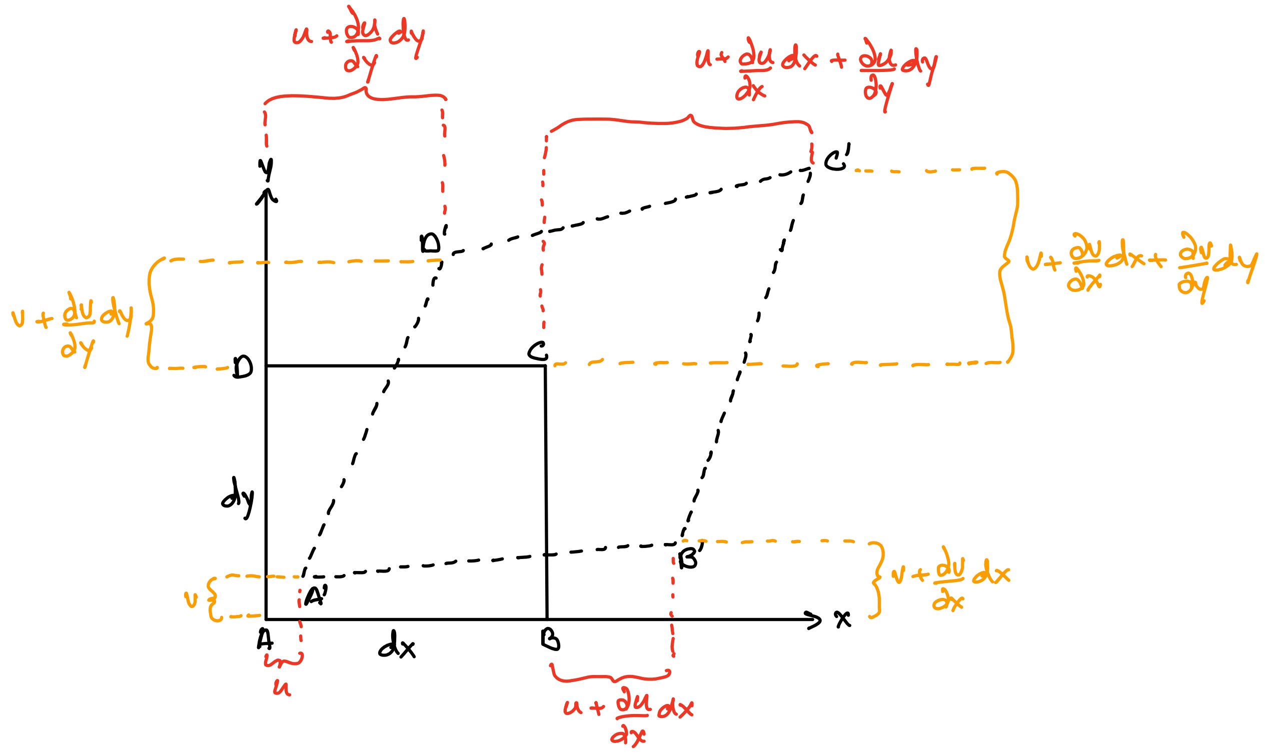
Prior to loading, the plate below has dimensions 1m x 0.8m. Under load, the plate deforms as shown below. Determine the components of strain at point A.



$$\varepsilon_x = \frac{u_B - u_A}{l_x} = \frac{1.6mm - 1.8mm}{1000mm} = -0.2 \times 10^{-3}$$

$$\varepsilon_y = \frac{v_D - v_A}{l_y} = \frac{-0.8mm - 0mm}{800mm} = -1 \times 10^{-3}$$

$$\gamma_{xy} = \frac{u_D - u_A}{l_y} + \frac{v_B - v_A}{l_x} = \frac{0mm - 1.8mm}{800mm} + \frac{2mm - 0mm}{1000mm} = -2.5 \times 10^{-4}$$

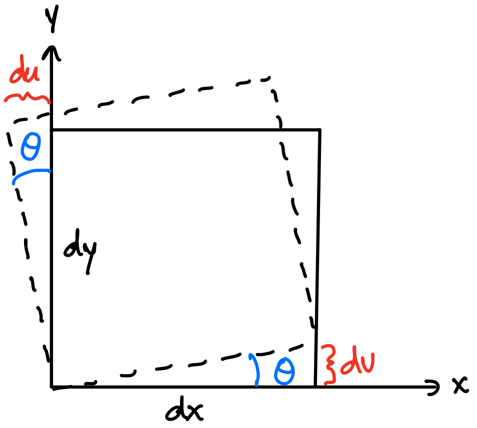


$$u(x,y) = u(0,0) + \frac{\partial u}{\partial x}\bigg|_{(0,0)} dx + \frac{\partial u}{\partial y}\bigg|_{(0,0)} dy \longrightarrow u_c = u_A + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

similar for $v(x,y)$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \varepsilon_x dx + \frac{1}{2}\frac{\partial u}{\partial y} dy + \frac{1}{2}\frac{\partial u}{\partial y} dy$$
$$= \varepsilon_x dx + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dy + \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dy$$
$$= \varepsilon_x dx + \frac{1}{2}\gamma_{xy} dy - \omega_z dy$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \frac{1}{2}\frac{\partial v}{\partial x} dx + \frac{1}{2}\frac{\partial v}{\partial x} dx + \varepsilon_y dy$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx + \varepsilon_y dy$$
$$= \varepsilon_y dy + \frac{1}{2}\gamma_{xy} dx + \omega_z dx$$



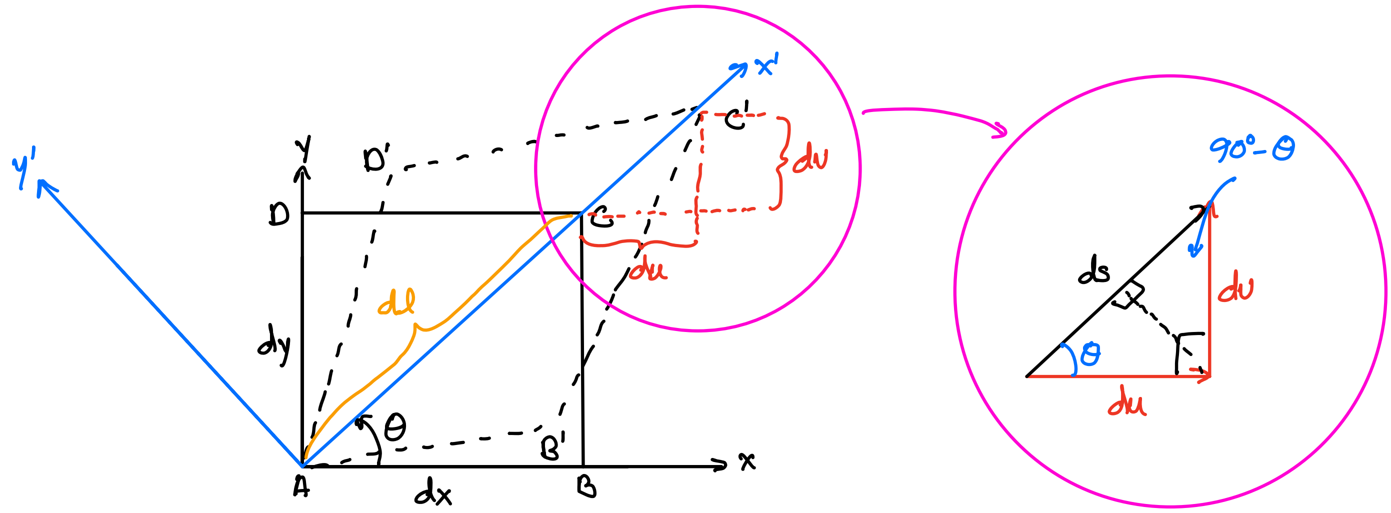
$$\tan\theta = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \approx \theta$$

$$\theta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \omega_z$$

In general, displacements at a point P are of the form:

$$u_p = u_0 + \varepsilon_x dx + \frac{1}{2}\gamma_{xy} dy - \omega_z dy$$

which involve $\underbrace{\text{rigid translations, rigid rotations}}_{\text{rigid body motion}}$, $\underbrace{\text{normal strains, and shear strains}}_{\text{deformation}}$.



$$ds = du\cos\theta + dv\sin\theta$$

$$ds = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\cos\theta + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\sin\theta$$

$$\varepsilon_{x'} = \frac{ds}{dl} = \left(\frac{\partial u}{\partial x}\frac{dx}{dl} + \frac{\partial u}{\partial y}\frac{dy}{dl}\right)\cos\theta + \left(\frac{\partial v}{\partial x}\frac{dx}{dl} + \frac{\partial v}{\partial y}\frac{dy}{dl}\right)\sin\theta$$

$$(dl)^2 = (dx)^2 + (dy)^2$$

$$= \varepsilon_x\cos^2\theta + \frac{\partial u}{\partial y}\sin\theta\cos\theta + \frac{\partial v}{\partial x}\sin\theta\cos\theta + \varepsilon_y\sin^2\theta$$

$$= \varepsilon_x\cos^2\theta + \varepsilon_y\sin^2\theta + \frac{\gamma_{xy}}{2}(2\sin\theta\cos\theta)$$

$$\left.\begin{aligned}
\varepsilon_{x'} &= \frac{1}{2}(\varepsilon_x+\varepsilon_y) + \frac{1}{2}(\varepsilon_x-\varepsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \\
\varepsilon_{y'} &= \frac{1}{2}(\varepsilon_x+\varepsilon_y) - \frac{1}{2}(\varepsilon_x-\varepsilon_y)\cos 2\theta - \frac{1}{2}\gamma_{xy}\sin 2\theta \\
\gamma_{x'y'} &= -(\varepsilon_x-\varepsilon_y)\sin 2\theta + \gamma_{xy}\cos 2\theta
\end{aligned}\right\}$$

The strain state in the new coord. sys. is related to the strain state in the old coord. sys.

These 2D strain transformation equations have the same form as the 2D stress transformation equations: $\sigma \longleftrightarrow \varepsilon$ and $\tau \longleftrightarrow \frac{1}{2}\gamma$.

$$\sigma_{x'} = \frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

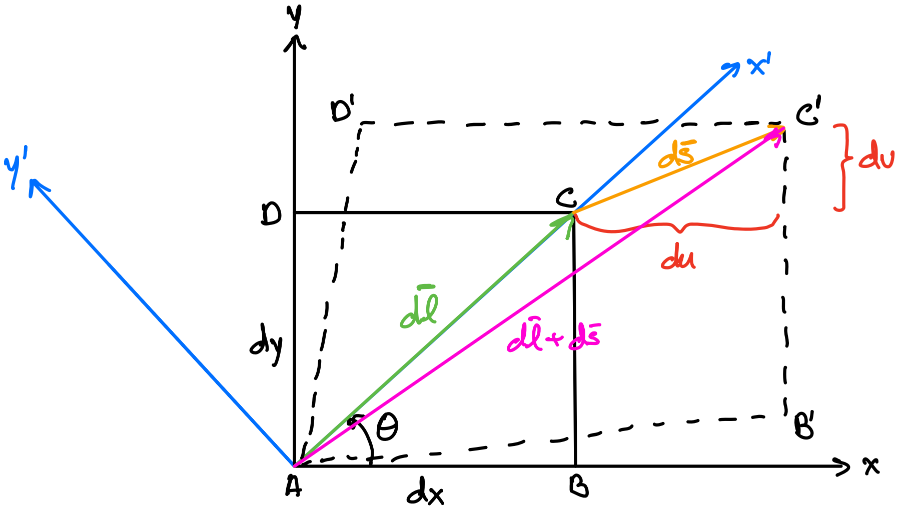
$$\sigma_{y'} = \frac{1}{2}(\sigma_x+\sigma_y) - \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\varepsilon_{x'} = \frac{1}{2}(\varepsilon_x+\varepsilon_y) + \frac{1}{2}(\varepsilon_x-\varepsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{1}{2}(\varepsilon_x+\varepsilon_y) - \frac{1}{2}(\varepsilon_x-\varepsilon_y)\cos 2\theta - \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\frac{1}{2}\gamma_{x'y'} = -\frac{1}{2}(\varepsilon_x-\varepsilon_y)\sin 2\theta + \frac{1}{2}\gamma_{xy}\cos 2\theta$$



$\hat{n}_{x'} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

note: $d\bar{l}$ ↑ vector dl ↑ magnitude

$u_c = u_A + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ $du = u_c - u_A$ $v_c = v_A + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial x}dx$ $dv = v_c - v_A$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$d\bar{s} = \begin{bmatrix} du \\ dv \end{bmatrix}$ (deformation in {x,y} coord. sys.)

$\frac{dx}{dl} = \cos\theta$ $\frac{dy}{dl} = \sin\theta$

$$\underbrace{\frac{d\bar{s}}{dl}} = \frac{1}{dl}\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}\frac{dx}{dl} + \frac{\partial u}{\partial y}\frac{dy}{dl} \\ \frac{\partial v}{\partial x}\frac{dx}{dl} + \frac{\partial v}{\partial y}\frac{dy}{dl} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \underbrace{\mathcal{E}\hat{n}_{x'}}$$

(strain in {x',y'} coord. sys.)

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial x}$$
$$= \frac{1}{2}\gamma_{xy} - \omega_z$$

$$\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial y}$$
$$= \frac{1}{2}\gamma_{xy} + \omega_z$$

we are only interested in deformation, so we ignore ω_z in the transformation. It's not relevant.

Project the strain onto the x'-axis (normal strain):

$\varepsilon_{x'} = \hat{n}^T_{x'}\,\mathcal{E}\,\hat{n}_{x'}$

Project the strain onto the y'-axis (shear strain):

$\frac{1}{2}\gamma_{x'y'} = \hat{n}^T_{y'}\,\mathcal{E}\,\hat{n}_{x'}$ note the transpose: $\frac{1}{2}\gamma_{x'y'} = (\hat{n}^T_{y'}\,\mathcal{E}\,\hat{n}_{x'})^T = \hat{n}^T_{x'}\,\mathcal{E}\,\hat{n}_{y'}$

In general: $\mathcal{E}' = \begin{bmatrix} \varepsilon_{x'} & \frac{1}{2}\gamma_{x'y'} \\ \frac{1}{2}\gamma_{x'y'} & \varepsilon_{y'} \end{bmatrix} = N^T \mathcal{E} N$

Can we define an axis such that, when the material deforms, the is no change in angle (i.e., no shear strain; only normal strain)?

$$d\bar{s} = ds\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = ds\,\hat{n} \qquad \text{displacement}$$

$$\frac{d\bar{s}}{dl} = \mathcal{E}\hat{n} \qquad \text{strain "traction"}$$

$$\mathcal{E}\hat{n}_p - \varepsilon_p\hat{n}_p = 0 \longrightarrow (\mathcal{E} - \varepsilon_p I)\hat{n}_p = 0$$

$$|\mathcal{E} - \varepsilon_p I| = \varepsilon_p^3 - I_1\varepsilon_p^2 + I_2\varepsilon_p - I_3 = 0$$

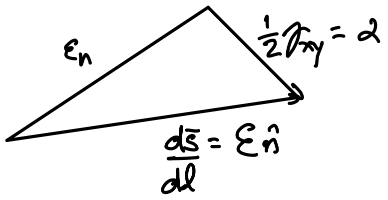
$$I_1 = tr(\mathcal{E})\ ;\ I_2 = M_{11} + M_{22} + M_{33}\ ;\ I_3 = |\mathcal{E}|$$

Once the principal strains are known, the strains acting on an inclined plane can be found using $\mathcal{E}' = N^T\mathcal{E}N$.

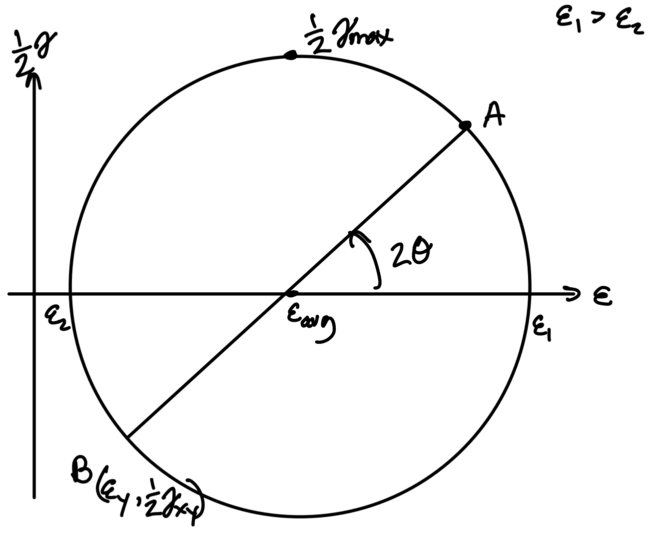
$$\varepsilon_n = \hat{n}^T\mathcal{E}\hat{n} = \varepsilon_1 n_x^2 + \varepsilon_2 n_y^2 + \varepsilon_3 n_z^2$$

$$d^2 = \varepsilon_1^2 n_x^2 + \varepsilon_2^2 n_y^2 + \varepsilon_3^2 n_z^2$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$



$\varepsilon_1 > \varepsilon_2 > \varepsilon_3$



1. locate $\varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$
2. locate A $(\varepsilon_x, -\frac{1}{2}\gamma_{xy})$
3. locate B $(\varepsilon_y, \frac{1}{2}\gamma_{xy})$

With the strain-displacement relationships, if the displacements are given in a problem, then we can determine the strains; if the strains are given, then the unknown displacements can determined. For the latter, in 2D, there are three equations and two unknowns. If the displacements are given, we can determine the corresponding strains; however, if the strains are given, we cannot necessarily determine the related displacements.

Determine the displacements from the position-dependent strains:

$$\varepsilon_x = \frac{\partial u}{\partial x} = y \qquad \varepsilon_y = \frac{\partial v}{\partial y} = y \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = xy$$

From ε_x and ε_y:

$$u = \int y\,dx = xy + c_1(y) \qquad v = \int y\,dy = \frac{y^2}{2} + c_2(x)$$

But these displacements are incompatible with the known shear strain:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x + c_1'(y) + c_2'(x) \neq xy$$

These displacements are even incompatible with each other, which we see if we try to calculate u from v and γ_{xy}:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + c_2'(x) = xy \qquad \therefore \qquad \frac{\partial u}{\partial y} = xy - c_2'(x)$$

$$u = \int (xy - c_2'(x))\,dy = \frac{1}{2}xy^2 - c_2'(x)y + c_3(x) \neq xy + c_1(y)$$

$$\left.\begin{aligned} \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \frac{\partial^2 \varepsilon_x}{\partial z} + \frac{\partial^2 \varepsilon_z}{\partial x^2} &= \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \\ \frac{\partial^2 \varepsilon_y}{\partial z} + \frac{\partial^2 \varepsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \end{aligned}\right\}$$

(2D) compatibility equations ensure continuity by ensuring displacements are single-valued

Given the displacement field and strain distribution,

$u = a_0x^2y^2 + a_1xy^2 + a_2x^2y$ $\qquad v = b_0x^2y + b_1xy$ $\qquad \gamma_{xy} = c_0x^2y$

what is the relationship among a_i, b_i, and c_i that make the definitions possible?

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2a_0xy^2 + a_1y^2 + 2a_2xy$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = b_0x^2 + b_1x$$

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y} \longrightarrow 4a_0x + 2a_1 + 2b_0 = 2c_0x$$

$$2\underbrace{(2a_0 - c_0)}_{=0}x + 2\underbrace{(a_1 + b_0)}_{=0} = 0 \qquad \therefore \boxed{c_0 = 2a_0 \;;\; a_1 = -b_0}$$

$$\therefore \quad u = a_0x^2y^2 - b_0xy^2 + a_2x^2y \qquad v = b_0x^2y + b_1xy \qquad \gamma_{xy} = 2a_0x^2y$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= 2a_0x^2y - 2b_0xy + a_2x^2 \\ \frac{\partial v}{\partial x} &= 2b_0xy + b_1y \end{aligned}\right\} \begin{aligned} &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2a_0x^2y + a_2x^2 + b_1y = 2a_0x^2y \\ &b_1y = -a_2x^2 \qquad \therefore \boxed{b_1 = a_2 = 0} \end{aligned}$$

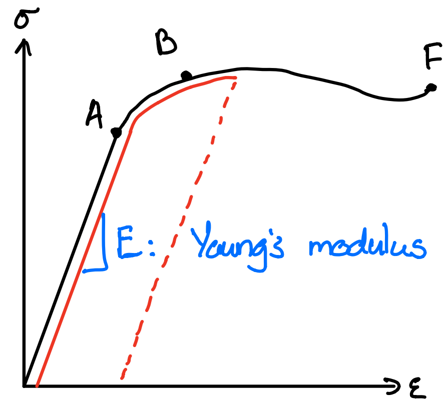
$$\therefore \; u = a_0x^2y^2 - b_0xy^2 \qquad v = b_0x^2y \qquad \gamma_{xy} = 2a_0x^2y$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2a_0xy^2 - b_0y^2 \qquad \varepsilon_y = \frac{\partial v}{\partial y} = b_0x^2$$

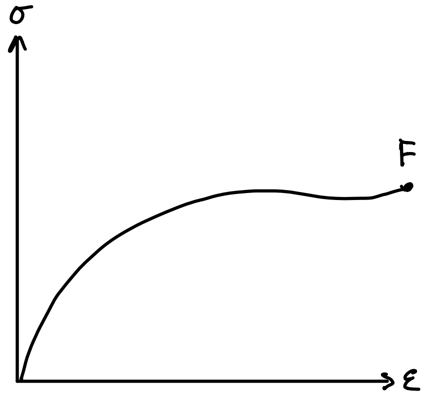
TOPIC 4:
Constitutive Relations

In our study of stress, we deal with the forces on a body. In our study of strain, we deal with the geometric changes of a body. In each case, we made no reference to the material of the body, so those formulas are valid for all materials. But we know from experience that stress and strain are related to each other — these are the constitutive equations.

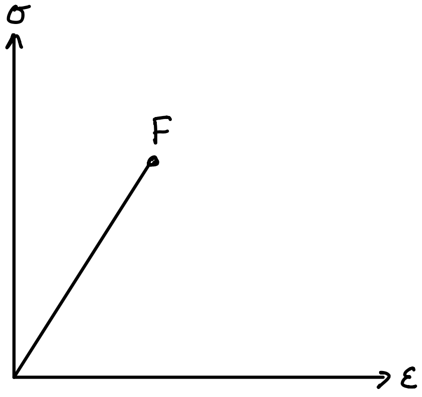
Most Engineering Materials



Elastomeric (rubbery) Material



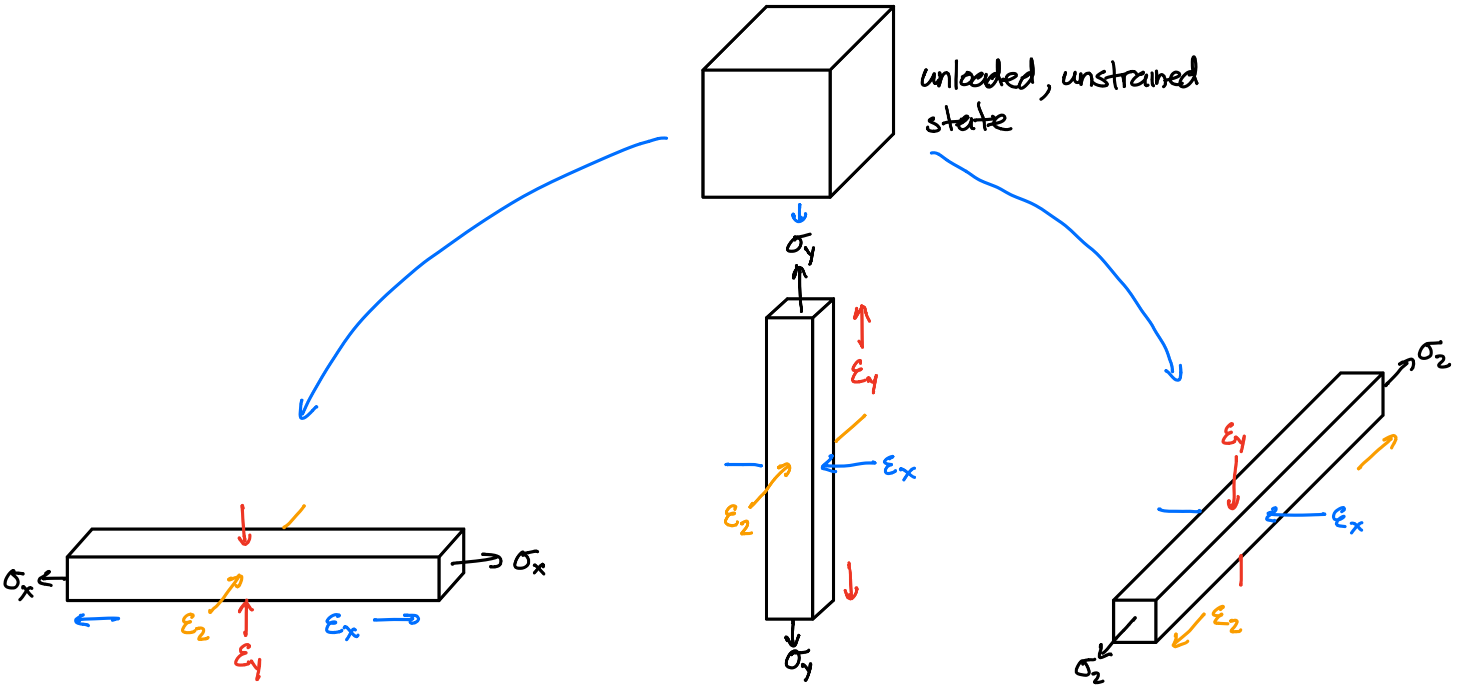
Ceramic (brittle) Material



(A) Linear limit: stress is a linear function of strain ($\sigma = E\varepsilon$) up to this point. Beyond this point, non-linear effect emerge.

(B) Elastic limit: if the material specimen is loaded below this limit and then released, then the material specimen returns to its unstrained state ($\varepsilon = 0$). If, however, the material specimen is loaded beyond this limit and then unloaded, then the material specimen does not return to its undeformed state ($\varepsilon \neq 0$).

(F) Failure: Ultimately, a material specimen cannot handle more loading and ruptures/cracks.



loading with σ_x yields ε_x, but also ε_y and ε_z

loading with σ_y yields ε_y, but also ε_x and ε_z

loading with σ_z yields ε_z, but also ε_x and ε_y

Poissen's ratio, ν, accounts for the effect where an expansion in one direction leads to a contraction in the perpendicular direction.

Loading by σ_x

$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\varepsilon_y = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

$$\varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

Loading by σ_y

$$\varepsilon_y = \frac{\sigma_y}{E}$$

$$\varepsilon_x = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E}$$

$$\varepsilon_z = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E}$$

Loading by σ

$$\varepsilon_z = \frac{\sigma_z}{E}$$

$$\varepsilon_x = -\nu\varepsilon_z = -\nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\varepsilon_z = -\nu\frac{\sigma_z}{E}$$

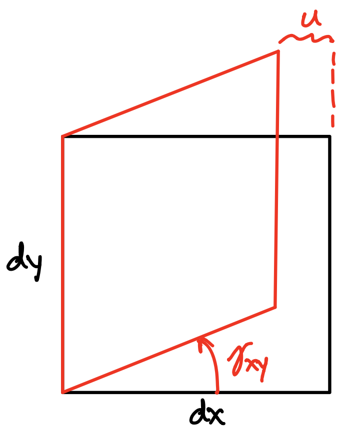
$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \qquad \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \qquad \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

In terms of $\sigma(\varepsilon)$:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$



$$du = \varepsilon_x dx = (1 - \underbrace{\cos\gamma_{xy}}_{\cos\gamma_{xy} \approx 1 \text{ since } \gamma_{xy} \text{ is small}})dx \approx 0$$

$\therefore \varepsilon_x \approx 0$, normal strains do not affect shear strains

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

3D Isotropic Constitutive Relations:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}2G+\lambda & \lambda & \lambda & 0 & 0 & 0\\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0\\ 0 & 0 & 0 & G & 0 & 0\\ 0 & 0 & 0 & 0 & G & 0\\ 0 & 0 & 0 & 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}$$

Young's modulus (E): measures material resistance to normal loads.

Shear modulus (G): measures material resistance to shear loads.

Bulk modulus (K): measures material resistance to hydrostatic loads.

Lamé's 1st parameter (λ): no physical interpretation. Simplifies elasticity matrix.

$$= \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}$$

$$G = \frac{E}{2(1+\nu)} \qquad K = \frac{E}{3(1-2\nu)}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$0 < E < \infty$

$-1 < \nu < 1/2$

Problems involving the full 3D constitutive equations are often difficult to solve without the use of numerical tools. Conditions of plane stress and plane strain allow the analysis to be simplified and applied to 2D problems.

plane stress: one dimension much smaller than the other two: thin plate, thin walled pressure vessel, thin disk, gears. All loading is in the plane.

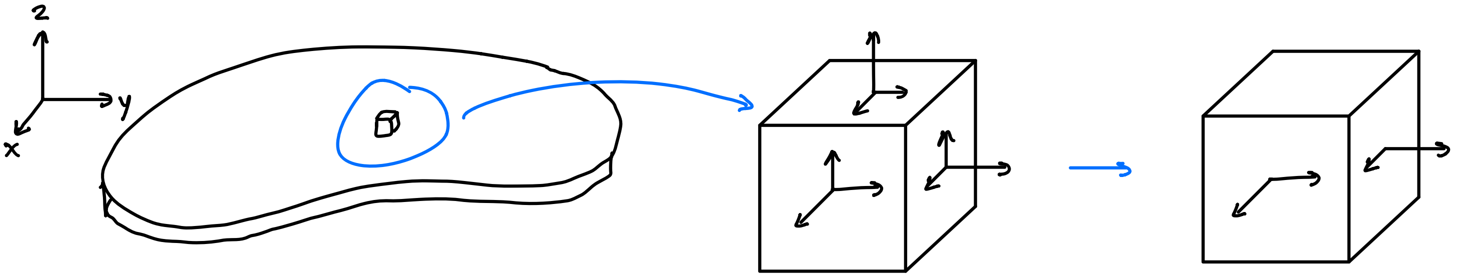
$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \tau_{xy} = G\gamma_{xy}$$

$$\longrightarrow \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}\frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0\\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0\\ 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_x + \sigma_y) \qquad \gamma_{xz} = \gamma_{yz} = 0 \qquad \sigma_z = \tau_{xz} = \tau_{yz} = 0$$



Constitutive Equations:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \qquad \varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

Plane Strain Conditions: $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \qquad \cancelto{0}{\varepsilon_z} = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\therefore \quad \sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \cancelto{0}{\gamma_{xz}} = \frac{1}{G}\tau_{xz} \qquad \cancelto{0}{\gamma_{yz}} = \frac{1}{G}\tau_{yz}$$

$$\therefore \quad \tau_{xz} = 0 \qquad \therefore \quad \tau_{yz} = 0$$

Substitute $\sigma_z = \nu(\sigma_x + \sigma_y)$ into ε_x and ε_y:

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu[\sigma_y + \nu(\sigma_x + \sigma_y)]) = \frac{1-\nu^2}{E}\sigma_x - \frac{(1+\nu)\nu}{E}\sigma_y \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu[\sigma_x + \nu(\sigma_x + \sigma_y)]) = -\frac{(1+\nu)\nu}{E}\sigma_x + \frac{1-\nu^2}{E}\sigma_y \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy}
\end{aligned}\right\}
\xrightarrow[\sigma_x,\sigma_y,\text{ and }\tau_{xy}]{\text{solve for}}
\left\{\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \quad (1) \\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu\varepsilon_x] \quad (2) \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}\right.$$

Matrix Form:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

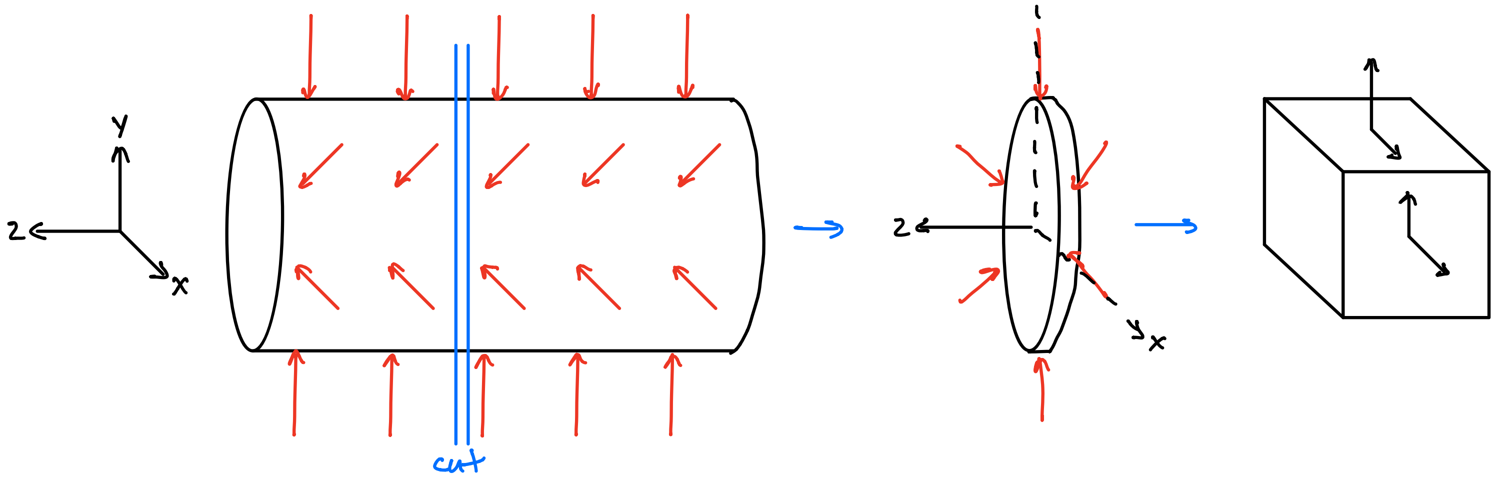
Substitute (1) and (2) into σ_z:

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y)$$

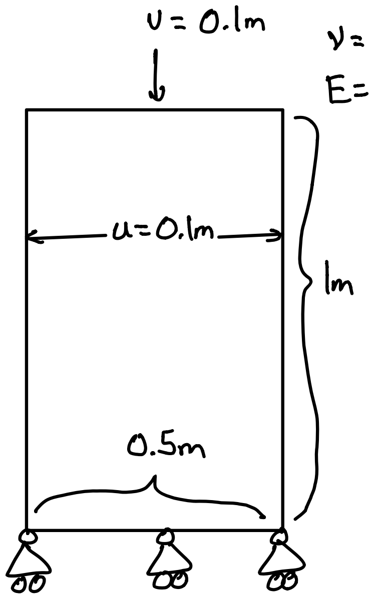
plane strain: one dimension much larger than the other two: tubes shafts under pressure, retaining walls or dams. Loading is constant (or nearly so) in the extended dimension. The strain in the extended dimension insignificant.

$$\varepsilon_x = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) \qquad \varepsilon_y = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$$\left.\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y\right]\\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\varepsilon_x\right]\\
\tau_{xy} &= G\gamma_{xy}\\
\sigma_z &= \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y) \qquad \tau_{xz} = \tau_{yz} = 0
\end{aligned}\right\} \rightarrow \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$$

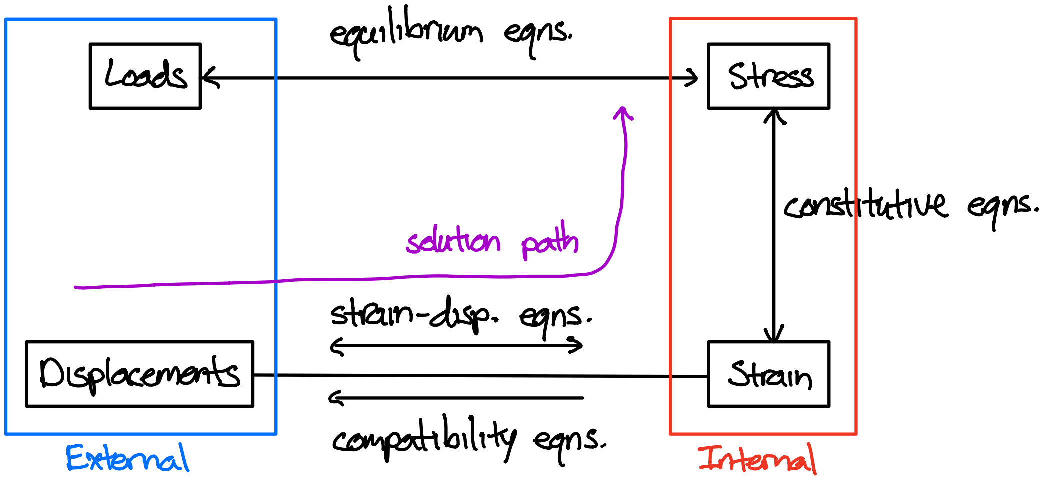


Given the thin plate with the dimensions and material properties shown below, in response to applied displacements, what are the internal stresses?



$\nu = 0.3$

$E = 91kPa$



$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{0.1\text{m}}{0.5\text{m}} = 0.2$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\frac{0.1\text{m}}{1.0\text{m}} = -0.1$$

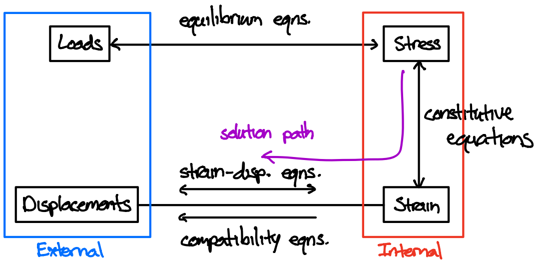
plane stress

$$\therefore \sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = 17\text{kPa}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = -4\text{kPa}$$

Given the following stress field with constants a and b, determine whether the stress distribution represent a solution for a plane strain problem.

$$S = \begin{bmatrix} a[y^2 + b(x^2-y^2)] & -2abxy & 0 \\ -2abxy & a[x^2 - b(x^2-y^2)] & 0 \\ 0 & 0 & ab(x^2+y^2) \end{bmatrix}$$



First, we must verify that S represents an equilibrium stress state:

$$\left.\begin{aligned} \sum F_x &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 2abx - 2abx + 0 = 0 \\ \sum F_y &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 2aby - 2aby + 0 = 0 \\ \sum F_z &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 + 0 + 0 = 0 \end{aligned}\right\} \text{equilibrium verified } \checkmark$$

We apply the constitutive equations for plane strain:

$$\varepsilon_x = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) \qquad \varepsilon_y = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \text{constitutive equations}$$

If we wish to determine the displacement field, we must first verify compatibility:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\frac{2a(1+\nu)(b+\nu-1)}{G} \neq 0$$

compatibility violated unless $a=0$,
but this would mean $S=0$.

So far, we have written constitutive relations for the special case of an isotropic material; however, other constitutive relations are possible. The isotropic relations come with certain symmetry assumptions. Let's consider the most general case (no assumptions) of an anisotropic material

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{bmatrix}$$

each c_{ij} is a material property (e.g., E, G, ν) or combination of properties which must be determined from mechanical testing... a lot of work!

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \\ \tau_{yx} \\ \tau_{zx} \\ \tau_{zy} \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} & c_{19} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} & c_{29} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} & c_{39} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} & c_{49} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & c_{57} & c_{58} & c_{59} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & c_{67} & c_{68} & c_{69} \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & c_{77} & c_{78} & c_{79} \\ c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & c_{88} & c_{89} \\ c_{91} & c_{92} & c_{93} & c_{94} & c_{95} & c_{96} & c_{97} & c_{98} & c_{99} \end{bmatrix}}_{\text{81 independent material constants}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ \gamma_{yx} \\ \gamma_{zx} \\ \gamma_{zy} \end{bmatrix}$$

We know that, for equilibrium, the off-diagonal shear stresses must be equal to each other (e.g., $\tau_{xy} = \tau_{yx}$) so that the stress state, S, is a symmetric tensor. The strain state, ε, is also symmetric. This means that the τ_{yz}, τ_{xz}, and τ_{xy} equation are identical to the τ_{yx}, τ_{zx}, and τ_{zy} equations. Therefore we can get rid of the three "extra" equations.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \\ \cancel{\tau_{yx}} \\ \cancel{\tau_{zx}} \\ \cancel{\tau_{zy}} \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & \cancel{c_{17}} & \cancel{c_{18}} & \cancel{c_{19}} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & \cancel{c_{27}} & \cancel{c_{28}} & \cancel{c_{29}} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & \cancel{c_{37}} & \cancel{c_{38}} & \cancel{c_{39}} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & \cancel{c_{47}} & \cancel{c_{48}} & \cancel{c_{49}} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & \cancel{c_{57}} & \cancel{c_{58}} & \cancel{c_{59}} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & \cancel{c_{67}} & \cancel{c_{68}} & \cancel{c_{69}} \\ \cancel{c_{71}} & \cancel{c_{72}} & \cancel{c_{73}} & \cancel{c_{74}} & \cancel{c_{75}} & \cancel{c_{76}} & \cancel{c_{77}} & \cancel{c_{78}} & \cancel{c_{79}} \\ \cancel{c_{81}} & \cancel{c_{82}} & \cancel{c_{83}} & \cancel{c_{84}} & \cancel{c_{85}} & \cancel{c_{86}} & \cancel{c_{87}} & \cancel{c_{88}} & \cancel{c_{89}} \\ \cancel{c_{91}} & \cancel{c_{92}} & \cancel{c_{93}} & \cancel{c_{94}} & \cancel{c_{95}} & \cancel{c_{96}} & \cancel{c_{97}} & \cancel{c_{98}} & \cancel{c_{99}} \end{bmatrix}}_{\text{81 independent material constants}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ \cancel{\gamma_{yx}} \\ \cancel{\gamma_{zx}} \\ \cancel{\gamma_{zy}} \end{bmatrix} \rightarrow$$

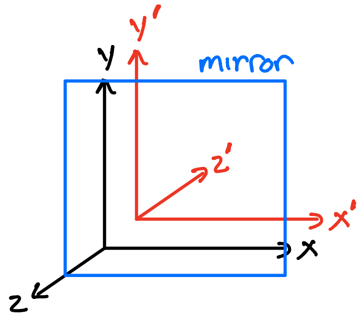
$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix}}_{\text{36 independent material constants}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

From energy considerations, the off-diagonal material constants must equal (e.g., $C_{12} = C_{21}$). This further reduces the number of independent constants.

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}}_{\text{36 independent material constants}} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix} \rightarrow \begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{bmatrix}}_{\text{21 independent material constants}} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}$$

plane of symmetry: material response is the same in the original and mirrored coordinate system.

Let the z-plane be a mirror plane:



$$x' = x \qquad y' = y \qquad z' = -z$$

$$\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}}_{N} \begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$$S' = N^T S N = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} \begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

notice we are using the engineering shear strain

$$\varepsilon' = N^T \varepsilon N = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} \begin{bmatrix}\varepsilon_x & \boxed{\gamma_{xy}} & \gamma_{xz}\\ \gamma_{xy} & \varepsilon_y & \gamma_{yz}\\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z\end{bmatrix} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$\sigma_{x'} = \sigma_x \qquad \sigma_{y'} = \sigma_y \qquad \sigma_{z'} = \sigma_z \qquad \tau_{y'z'} = -\tau_{yz} \qquad \tau_{x'z'} = -\tau_{xz} \qquad \tau_{x'y'} = \tau_{xy}$$

$$\varepsilon_{x'} = \varepsilon_x \qquad \varepsilon_{y'} = \varepsilon_y \qquad \varepsilon_{z'} = \varepsilon_z \qquad \gamma_{y'z'} = -\gamma_{yz} \qquad \gamma_{x'z'} = -\gamma_{xz} \qquad \gamma_{x'y'} = \gamma_{xy}$$

original coordinates, fully anisotropic material

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}$$

primed coordinates

$$\begin{bmatrix}\sigma_x'\\ \sigma_y'\\ \sigma_z'\\ \tau_{yz}'\\ \tau_{xz}'\\ \tau_{xy}'\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_x'\\ \varepsilon_y'\\ \varepsilon_z'\\ \gamma_{yz}'\\ \gamma_{xz}'\\ \gamma_{xy}'\end{bmatrix}$$

{substitute primed stresses/strains for the original terms}

transfer (-) signs to the elastic coefficients, c_{ij}

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ -\tau_{yz}\\ -\tau_{xz}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ -\gamma_{yz}\\ -\gamma_{xz}\\ \gamma_{xy}\end{bmatrix}$$

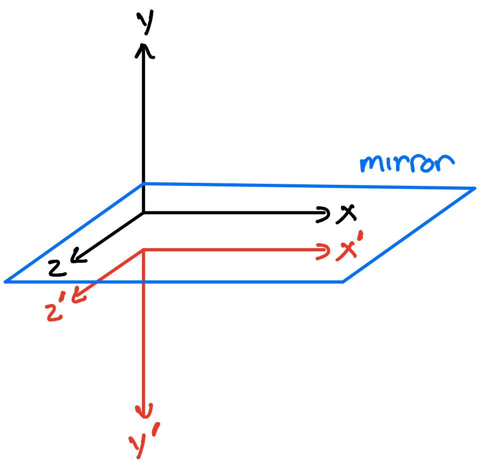
$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \begin{array}{c|cccccc|} & & & & (-) & (-) & \\ & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ & c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ & c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ (-) & c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ (-) & c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ & c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{array} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}$$

the material is the same in any coordinate system – you do not turn lead into gold by a coordinate transformation! Consequently, the components of the elasticity matrix in both the primed and original expressions must match. If $-c_{ij} = c_{ij}$, then $c_{ij} = 0$.

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16}\\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26}\\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36}\\ 0 & 0 & 0 & c_{44} & c_{45} & 0\\ 0 & 0 & 0 & c_{45} & c_{55} & 0\\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}$$

z-plane of symmetry

Let the y-plane be an additional mirror plane:



$$x' = x \qquad y' = -y \qquad z' = z$$

$$\begin{bmatrix}x'\\y'\\z'\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}}_{N}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\sigma_{x'} = \sigma_x \qquad \sigma_{y'} = \sigma_y \qquad \sigma_{z'} = \sigma_z \qquad \tau_{y'z'} = -\tau_{yz} \qquad \tau_{x'z'} = \tau_{xz} \qquad \tau_{x'y'} = -\tau_{xy}$$

$$\varepsilon_{x'} = \varepsilon_x \qquad \varepsilon_{y'} = \varepsilon_y \qquad \varepsilon_{z'} = \varepsilon_z \qquad \gamma_{y'z'} = -\gamma_{yz} \qquad \gamma_{x'z'} = \gamma_{xz} \qquad \gamma_{x'y'} = -\gamma_{xy}$$

original coordinates, z-plane of symmetry

$$\begin{bmatrix}\sigma_x\\\sigma_y\\\sigma_z\\\tau_{yz}\\\tau_{xz}\\\tau_{xy}\end{bmatrix} = \begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&c_{16}\\c_{12}&c_{22}&c_{23}&0&0&c_{26}\\c_{13}&c_{23}&c_{33}&0&0&c_{36}\\0&0&0&c_{44}&c_{45}&0\\0&0&0&c_{45}&c_{55}&0\\c_{16}&c_{26}&c_{36}&0&0&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\varepsilon_z\\\gamma_{yz}\\\gamma_{xz}\\\gamma_{xy}\end{bmatrix}$$

primed coordinates

$$\begin{bmatrix}\sigma_{x'}\\\sigma_{y'}\\\sigma_{z'}\\\tau_{y'z'}\\\tau_{x'z'}\\\tau_{x'y'}\end{bmatrix} = \begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&c_{16}\\c_{12}&c_{22}&c_{23}&0&0&c_{26}\\c_{13}&c_{23}&c_{33}&0&0&c_{36}\\0&0&0&c_{44}&c_{45}&0\\0&0&0&c_{45}&c_{55}&0\\c_{16}&c_{26}&c_{36}&0&0&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{x'}\\\varepsilon_{y'}\\\varepsilon_{z'}\\\gamma_{y'z'}\\\gamma_{x'z'}\\\gamma_{x'y'}\end{bmatrix}$$

{substitute primed stresses/strains for the original terms}

transfer (-) signs to the elastic coefficients, c_{ij}

$$\begin{bmatrix}\sigma_x\\\sigma_y\\\sigma_z\\-\tau_{yz}\\\tau_{xz}\\-\tau_{xy}\end{bmatrix} = \begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&c_{16}\\c_{12}&c_{22}&c_{23}&0&0&c_{26}\\c_{13}&c_{23}&c_{33}&0&0&c_{36}\\0&0&0&c_{44}&c_{45}&0\\0&0&0&c_{45}&c_{55}&0\\c_{16}&c_{26}&c_{36}&0&0&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\varepsilon_z\\-\gamma_{yz}\\\gamma_{xz}\\-\gamma_{xy}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_x\\\sigma_y\\\sigma_z\\\tau_{yz}\\\tau_{xz}\\\tau_{xy}\end{bmatrix} = \begin{array}{c|cccccc|} & & & & (-) & & (-) \\ \hline & c_{11}&c_{12}&c_{13}&0&0&c_{16}\\ & c_{12}&c_{22}&c_{23}&0&0&c_{26}\\ & c_{13}&c_{23}&c_{33}&0&0&c_{36}\\ (-) & 0&0&0&c_{44}&c_{45}&0\\ & 0&0&0&c_{45}&c_{55}&0\\ (-) & c_{16}&c_{26}&c_{36}&0&0&c_{66} \\ \hline \end{array}\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\varepsilon_z\\\gamma_{yz}\\\gamma_{xz}\\\gamma_{xy}\end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

both y- and z-planes of symmetry

Two planes of material symmetry imply a third, generating an orthotropic material. Notice the similar construction between the above and the isotropic constitutive equations.

TOPIC 5:
Problems in Elasticity, Part I: Airy’s Stress Function

Airy's Stress Function

From calculus: given two functions $f(x,y)$ and $g(x,y)$, if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$, then there is a third function $A(x,y)$ such that $f = \frac{\partial A}{\partial y}$ and $g = -\frac{\partial A}{\partial x}$.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \qquad \underbrace{\frac{\partial(\sigma_x - U)}{\partial x}}_{\frac{\partial f}{\partial x}} + \underbrace{\frac{\partial \tau_{xy}}{\partial y}}_{\frac{\partial g}{\partial y}} = 0 \quad \text{if } b_x = -\frac{\partial U}{\partial x}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0 \qquad \underbrace{\frac{\partial \tau_{xy}}{\partial x}}_{\frac{\partial f}{\partial x}} + \underbrace{\frac{\partial(\sigma_y - U)}{\partial y}}_{\frac{\partial g}{\partial y}} = 0 \quad \text{if } b_y = -\frac{\partial U}{\partial y}$$

2D equilibrium equations

$U = mgy$

$b_y F = -\frac{\partial U}{\partial y} = -mg$

From the first equation, using the calculus theorem and A as third function:

$$\sigma_x - U = \frac{\partial A}{\partial y} \qquad \tau_{xy} = -\frac{\partial A}{\partial x}$$

From the second equation, using the calculus theorem and B as third function:

$$\tau_{xy} = -\frac{\partial B}{\partial y} \qquad \sigma_y - U = \frac{\partial B}{\partial x}$$

Since $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} = 0$, then

$$f = \frac{\partial B}{\partial y} \;;\; g = -\frac{\partial B}{\partial x} \quad \text{or} \quad f = -\frac{\partial B}{\partial y} \;;\; g = \frac{\partial B}{\partial x}$$

Since τ_{xy} is identical:

$$\tau_{xy} = \tau_{xy} \longrightarrow \underbrace{\frac{\partial A}{\partial x} + \frac{\partial(-B)}{\partial y} = 0} \longrightarrow A = \frac{\partial \phi}{\partial y} \;;\; B = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}$$

→ this is just the calculus theorem all over again! with $f = A$ and $g = -B$. we choose ϕ (Airy stress function) as the third function.

$$\therefore\; \sigma_x - U = \frac{\partial A}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} \;;\; \sigma_y - U = \frac{\partial B}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} \;;\; \tau_{xy} = -\frac{\partial A}{\partial x} = -\frac{\partial B}{\partial y} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Above, ϕ only accounts for equilibrium eqns. Now involve the constitutive eqns. to go from stress to strain.

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right) + \frac{1-\nu}{E}U \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2} - \nu\frac{\partial^2\phi}{\partial y^2}\right) + \frac{1-\nu}{E}U \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = -\frac{2(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y}
\end{aligned}\right\} \text{plane stress}$$

Now, we substitute these strains into the compatibility eqn: $\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} - \frac{\partial^2\gamma_{xy}}{\partial x\partial y} = 0$

bi-harmonic equation

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} = -(1-\nu)\left[\frac{\partial^2U}{\partial x^2} + \frac{\partial^2U}{\partial y^2}\right] \quad \text{plane stress}$$

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} = \left(\frac{1-2\nu}{1-\nu}\right)\left[\frac{\partial^2U}{\partial x^2} + \frac{\partial^2U}{\partial y^2}\right] \quad \text{plane strain}$$

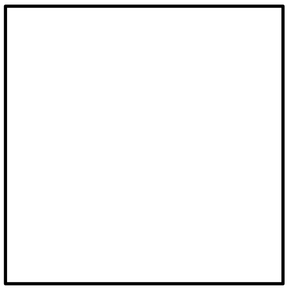
If $U=0$, then the plane stress and plane strain definitions converge:

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} = 0 \qquad (U=0)$$

Degree 1: $\phi = a_1x + a_2y$

$\nabla^4\phi = 0$

$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 0$ $\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 0$ $\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = 0$

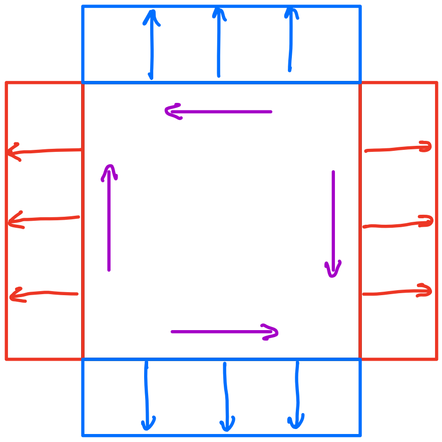


the trivial case where no load is applied

Degree 2: $\phi = a_1 x^2 + a_2 xy + a_3 y^2$

$\nabla^4 \phi = 0$

$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2a_3$ $\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 2a_1$ $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -a_2$

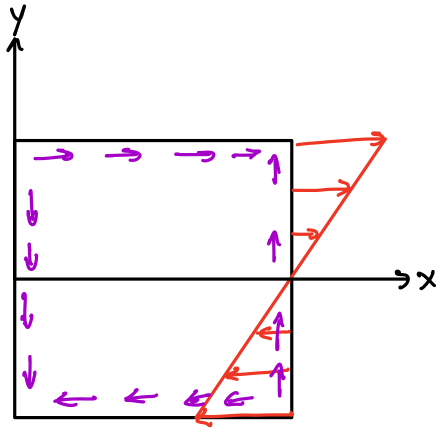


uniform loads and stress distributions
are represented

Degree 3: $\phi = a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3$

$\nabla^4 \phi = 0$

$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2a_3 x + 6a_4 y$ $\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 6a_1 x + 2a_2 y$ $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -2(a_2 x + a_3 y)$



$a_4 \neq 0$; $a_1 = a_2 = 0$

loads and stress distributions with
linear variability are represented

Degree 4: $\phi = a_1 x^4 + a_2 x^3 y + a_3 x^2 y^2 + a_4 x y^3 + a_5 y^4$

$\nabla^4 \phi = 8(3a_1 + a_3 + 3a_5) \neq 0$ coefficients must be related

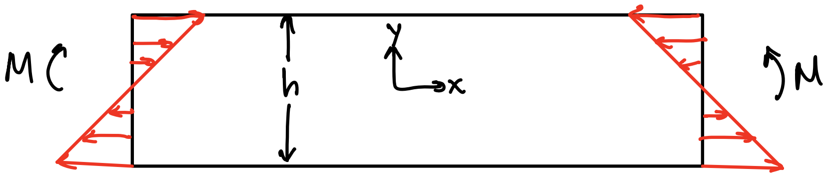
Let $a_3 = -3(a_1 + a_5)$ $\therefore$ $\phi = a_1 x^4 + a_2 x^3 y - 3(a_1 + a_5) x^2 y^2 + a_4 x y^3 + a_5 y^4$

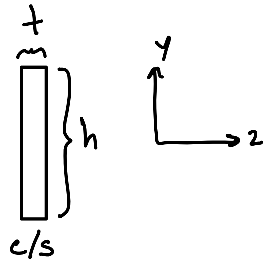
$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = -6(a_1+a_5)x^2 + 6a_4xy + 12a_5y^2$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 12a_1x^2 + 6a_2xy - 6(a_1+a_5)y^2$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -3a_2x^2 + 12(a_1+a_5)xy - 3a_4y^2$$

Determine the bending stress resulting from applying bending moment, M, to thin plate of unit thickness, t. Use Airy's stress function.





$$\phi = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 2a_3x + 6a_4y \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} = 6a_1x + 2a_2y \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -2(a_2x + a_3y)$$

The problem indicates that the plate is subjected to pure bending. This means no shear ($\tau_{xy}=0$) and $\sigma_y=0$. This information helps us eliminate some coefficients.

$a_1 = a_2 = a_3 = 0$ since $\tau_{xy}=0$ and $\sigma_y=0$ for all (x,y). Thus:

$$\sigma_x = 6a_4y$$

To determine a_4, we look to the loading, M. The moment due the bending stress must balance the loading.

$$M = \int(\sigma_x dA)y = \int_0^t\int_{-h/2}^{h/2} \sigma_x y\,dy\,dz = 6a_4t\int_{-h/2}^{h/2} y^2dy = \frac{1}{2}a_4th^3 \qquad \therefore \quad a_4 = \frac{2M}{th^3}$$

$$\sigma_x = 6a_4y = \frac{12M}{th^3}y = \frac{M}{I}y \qquad \therefore \quad I = \frac{th^3}{12}$$ area moment of inertia of rectangular cross-section

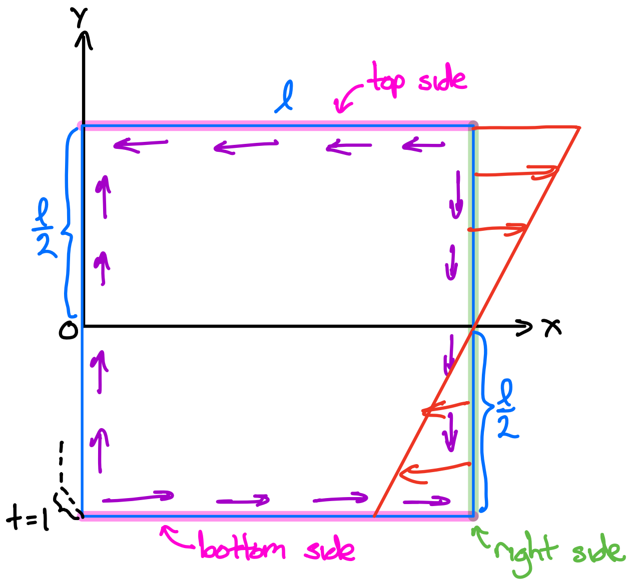
Show that the moments at the boundary, M^{σ} and M^{τ}, due to normal and shearing loads sum to zero in accordance with equilibrium requirements. Use a polynomial Airy's stress function of degree four.

$$\phi = a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4$$

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} = 24a_1 + 24a_5 + 8a_3 \qquad a_5 = -\frac{1}{3}(3a_1+a_3)$$

$$\phi = a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 - \frac{1}{3}(3a_1+a_3)y^4$$

$a_1 = a_2 = a_3 = 0;\ a_4 > 0$



$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 2a_3x^2 + 6a_4xy - 4(3a_1+a_3)y^2 = 6a_4xy$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 12a_1x^2 + 6a_2xy + 2a_3y^2 = 0$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -(3a_2x^2 + 4a_3xy + 3a_4y^2) = -3a_4y^2$$

$$M_0^{\sigma} = \int(\sigma_x dA)y = \int_0^{+}\int_{-l/2}^{l/2} \sigma_x y\, dydz = 6a_4 l t\int_{-l/2}^{l/2} y^2 dy = \frac{a_4 l^4 t}{2}$$

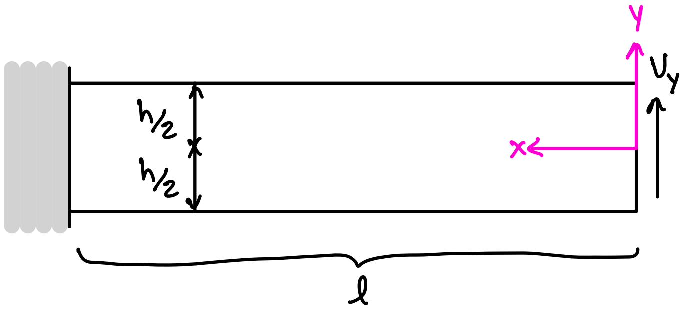
$$M_0^{\tau} = \underbrace{2\int(\tau_{xy}dA)\frac{l}{2}}_{\text{top-bottom}} + \underbrace{\int(\tau_{xy}dA)l}_{\text{right}} = -\frac{3}{4}a_4 l^3\int_0^{+}\int_0^{l} dxdz - 3a_4 l\int_0^{+}\int_{-l/2}^{l/2} y^2 dydz$$

$$= -\frac{3}{4}a_4 l^3 t\int_0^{l} dx - 3a_4 l t\int_{-l/2}^{l/2} y^2 dy$$

$$= -\frac{3}{4}a_4 l^4 t - \frac{1}{4}a_4 l^4 t = -\frac{a_4 l^4 t}{2}$$

$M^{\sigma} + M^{\tau} = 0 \quad \therefore$ equilibrum satisfied

Consider the cantilever beam below, which has a small thickness, t. Given the tip load, V_y, resulting from a shear load, determine the stress distribution and displacement via Airy's stress function. To start, assume a 4th-order polynomial, then add corrections as necessary.



$$\phi^4 = a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4$$

$$\nabla^4\phi^4 = 8(3a_1 + a_3 + 3a_5) = 0 \qquad \therefore \qquad a_5 = -\frac{1}{3}(3a_1 + a_3)$$

$$\phi^4 = a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 - \frac{1}{3}(3a_1 + a_3)y^4$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = -12a_1y^2 + 2a_3(x^2 - 2y^2) + 6a_4xy$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 12a_1x^2 + 6a_2xy + 2a_3y^2$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -(3a_2x^2 + 4a_3xy + 3a_4y^2)$$

The system with resultant tip load V_y is in shear; however, we also expect internal bending stress. No σ_y is applied or expected to result from the loading:

$$\sigma_y = 0 \qquad \therefore \qquad a_1 = a_2 = a_3 = 0$$

$$\sigma_x = 6a_4xy \qquad \sigma_y = 0 \qquad \tau_{xy} = -3a_4y^2$$

No shear is applied or expected to arise on the top/bottom surfaces due to the indicated loading; however, setting $a_4 = 0$ will result in $\phi^4 = 0$ and eliminate all stresses. We must consider an additional stress function:

$$\phi^2 = b_1x^2 + b_2xy + b_3y^2$$

$$\nabla^4\phi^2 = 0$$

$$\sigma_x = 2b_3 \qquad \sigma_y = 2b_1 \qquad \tau_{xy} = -b_2$$

$$\therefore \qquad \sigma_x = 6a_4xy + 2b_3 \qquad \sigma_y = 2b_1 \qquad \tau_{xy} = -(3a_4y^2 + b_2)$$

$\sigma_x = 0$ @ $y = 0$ (neutral axis) $\therefore$ $b_3 = 0$

$\sigma_y = 0$ @ $y = \pm\frac{h}{2}$ $\therefore$ $b_1 = 0$

$\tau_{xy} = 0$ @ $y = \pm\frac{h}{2}$ $\longrightarrow -\left[\frac{3a_4h^2}{4} + b_2\right] = 0$ $\therefore$ $a_4 = -\frac{4b_2}{3h^2}$

$$V_y = \int_A \tau_{xy}\,dA = b_2\int_0^{t}\int_{-h/2}^{+h/2}\left(\frac{4}{h^2}y^2 - 1\right)dy\,dz = -\frac{2htb_2}{3} \quad \therefore \quad b_2 = -\frac{3V_y}{2ht}$$

$$\sigma_x = -\frac{9V_y}{ht}xy \qquad \sigma_y = 0 \qquad \tau_{xy} = -\frac{3V_y}{2ht}\left(\frac{4}{h^2}y^2 - 1\right)$$

Let us now consider the displacement.

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -\frac{9V_y}{Eht}xy = \alpha xy$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{9V_y}{Eht}\nu xy = \beta xy$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2(1+\nu)}{E}\tau_{xy} = -\frac{3V_y(1+\nu)}{Eht}\left(\frac{4}{h^2}y^2 - 1\right) = \delta y^2 + \varepsilon$$

$$u = \int \varepsilon_x\,dx = \frac{\alpha}{2}x^2y + f(y) \qquad v = \int \varepsilon_y\,dy = \frac{\beta}{2}xy^2 + g(x)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\alpha}{2}x^2 + f'(y) + \frac{\beta}{2}y^2 + g'(x) = \delta y^2 + \varepsilon$$

$$\underbrace{g'(x) + \frac{\alpha}{2}x^2}_{\text{pure } x} + \underbrace{f'(y) + \left(\frac{\beta}{2} - \delta\right)y^2}_{\text{pure } y} = \varepsilon$$

we have two functions pure in x and y, respectively, that sum to a constant; therefore each function must be equal to a constant.

$$g'(x) + \frac{\alpha}{2}x^2 = C_1 \qquad f'(y) + \left(\frac{\beta}{2} - \delta\right)y^2 = C_2$$

$$g(x) = -\frac{\alpha}{6}x^3 + C_1x + C_3 \qquad f(y) = -\frac{1}{3}\left(\frac{\beta}{2} - \delta\right)y^3 + C_2y + C_4$$

$$\left.\begin{aligned} u &= \frac{\alpha}{2}x^2y - \frac{1}{3}\left(\frac{\beta}{2} - \delta\right)y^3 + C_2y + C_4 \\ v &= \frac{\beta}{2}xy^2 - \frac{\alpha}{6}x^3 + C_1x + C_3 \end{aligned}\right\}$$

displacement boundary conditions determine integration constants $C_1, \ldots, C_4$

$$\left.\begin{aligned} u(\ell,0) &= C_4 = 0 \\ v(\ell,0) &= -\frac{\alpha}{6}\ell^3 + C_1\ell + C_3 = 0 \end{aligned}\right\}$$

zero displacement at fixed boundary

$$\left.\frac{\partial v}{\partial x}\right|_{(\ell,0)} = -\frac{\alpha}{2}\ell^2 + C_1 = 0$$

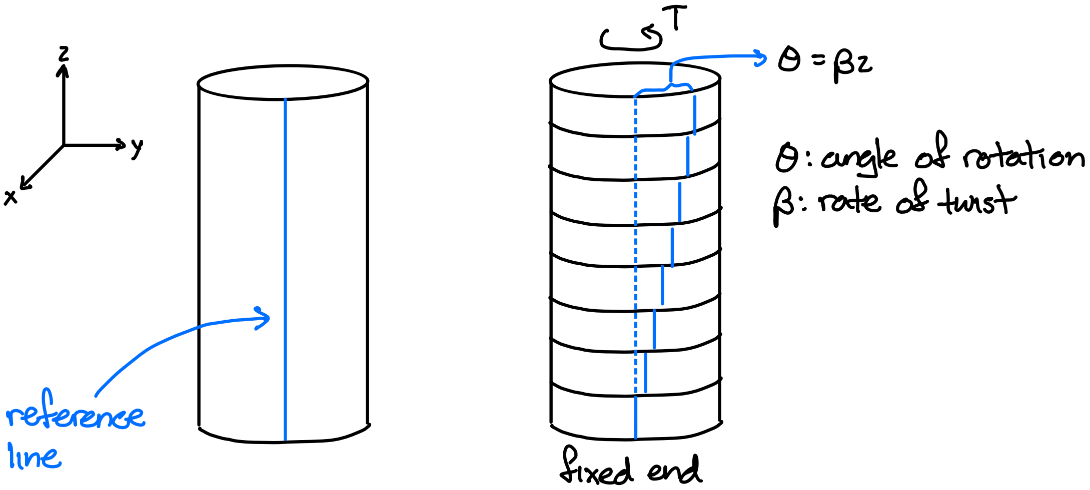
slope vanishes at fixed boundary

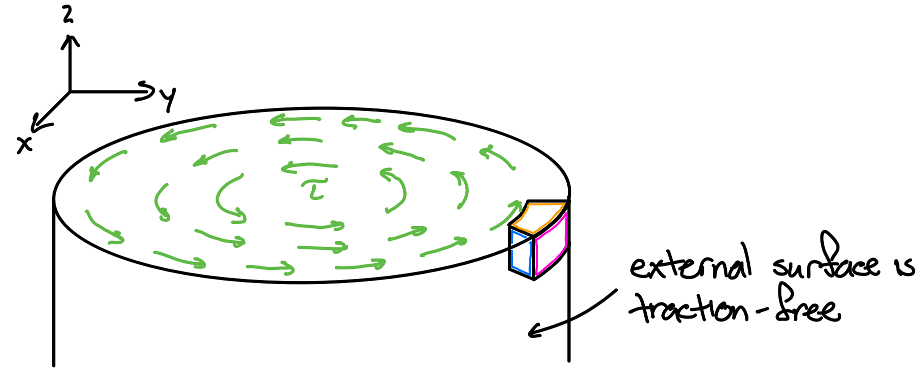
$$\left.\frac{\partial u}{dy}\right|_{(\ell,0)} = \frac{\alpha}{2}\ell^2 + C_2 = 0$$

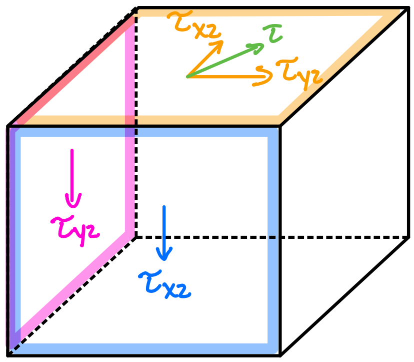
no rotation at fixed boundary

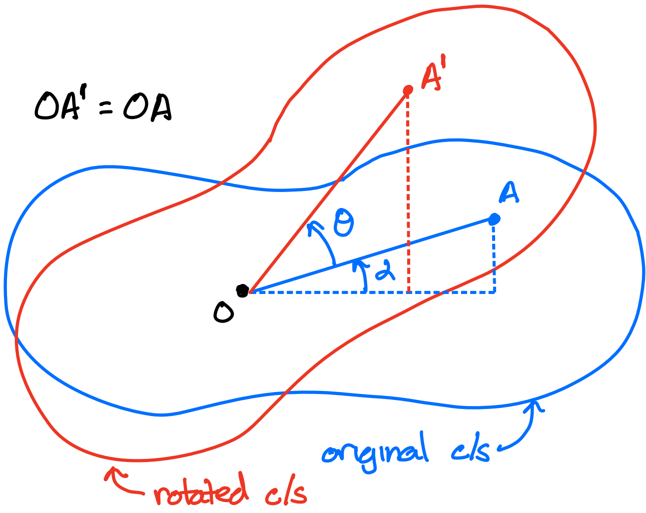
TOPIC 6:
Problems in Elasticity, Part II: Prandtl’s Stress Function

Saint-Venant's Theory of Torsion: of a prismatic element of arbitrary cross-section under torsional load, each cross-section rotates about the z-axis as a rigid body (i.e., no in-plane deformation/shape change) and all cross-sections deform (i.e., warp) out-of-plane in the same way. Also, the angle of rotation of each cross-section is linearly proportional to its position along the z-axis.









Every point in the cross-section rotates as a rigid body about the z-axis at point O (not necessarily the area centroid).

$$u = OA'\cos(\alpha+\theta) - OA\cos\alpha$$

$$= OA'\cos\alpha\cos\theta - OA'\sin\alpha\sin\theta - OA\cos\alpha$$

$$= OA'\cos\alpha - OA'\theta\sin\alpha - OA\cos\alpha = -\theta y = -\beta yz$$

$$v = OA'\sin(\alpha+\theta) - OA\sin\alpha$$

$$= OA'\sin\alpha\cos\theta + OA'\cos\alpha\sin\theta - OA\sin\alpha$$

$$= OA'\sin\alpha + OA'\theta\cos\alpha - OA\sin\alpha = \theta x = \beta xz$$

$w = \beta\psi(x,y)$ $\psi(x,y)$: warping function

From the above displacement field, we proceed through the Tonti diagram beginning with the calculation of the strains:

$$\varepsilon_x = \frac{du}{dx} = 0 \qquad \varepsilon_y = \frac{dv}{dy} = 0 \qquad \varepsilon_z = \frac{dw}{dz} = 0$$

$$\gamma_{xy} = \frac{du}{dy} + \frac{du}{dx} = -\beta z + \beta z = 0 \qquad \gamma_{xz} = \frac{du}{dz} + \frac{dw}{dx} = \beta\left(-y + \frac{d\psi}{dx}\right) \qquad \gamma_{yz} = \frac{dv}{dz} + \frac{dw}{dy} = \beta\left(x + \frac{d\psi}{dy}\right)$$

Inserting into the constitutive relations gives the stresses:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] = 0$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] = 0$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] = 0$$

$$\tau_{xy} = G\gamma_{xy} = 0 \qquad \tau_{xz} = G\gamma_{xz} = G\beta\left(-y + \frac{d\psi}{dx}\right) \qquad \tau_{yz} = G\gamma_{yz} = G\beta\left(x + \frac{d\psi}{dy}\right)$$

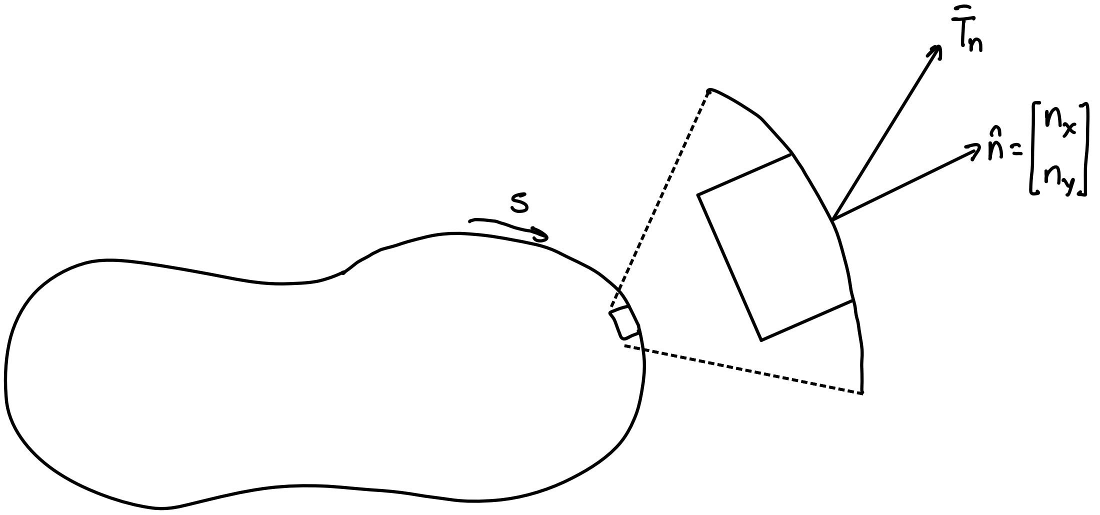
Verify that equilibrium is maintained:

$$\left.\begin{aligned} &\frac{d\sigma_x}{dx} + \frac{\partial \tau_{xy}}{dy} + \frac{\partial \tau_{xz}}{dz} + \cancelto{0}{b_x} = 0 \\ &\frac{\partial \tau_{xy}}{dx} + \frac{d\sigma_y}{dy} + \frac{\partial \tau_{yz}}{dz} + \cancelto{0}{b_y} = 0 \end{aligned}\right\} \text{automatically satisfied}$$

$$\frac{\partial \tau_{xz}}{dx} + \frac{\partial \tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + \cancelto{0}{b_z} = \boxed{\frac{\partial^2\psi}{dx^2} + \frac{\partial^2\psi}{dy^2} = 0}$$

solve Laplace's equation with boundary conditions to determine ψ and, by extension w

We've gone from displacement to strain (compatibility automatically satisfied), from strain to stress via the constitutive relations, and formulated Laplace's equation in ψ by enforcing equilibrium. The particular boundary conditions of the problem will determine the specific form of ψ.



traction-free surface

$$\bar{T}_n = \underbrace{\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}}_{\hat{n}} = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix} = G\beta\left(-y + \frac{\partial \psi}{\partial x}\right) n_x + G\beta\left(x + \frac{\partial \psi}{\partial y}\right) n_y = 0$$

$$= -y n_x + x n_y + \frac{\partial \psi}{\partial x} n_x + \frac{\partial \psi}{\partial y} n_y = 0$$

$$= -y n_x + x n_y + \frac{\partial \psi}{\partial n} = 0$$ difficult BC

$\nabla \psi = \frac{\partial \psi}{\partial x} \hat{i} + \frac{\partial \psi}{\partial y} \hat{j}$ gradient

$\nabla \psi \cdot \hat{n} = \frac{\partial \psi}{\partial x} n_x + \frac{\partial \psi}{\partial y} n_y = \frac{\partial \psi}{\partial n}$ projection of gradient onto normal axis

For an alternative expression of the BC, we return to the equilibrium equations:

$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$ just as in the case of Airy's stress function, we can define a function, ϕ

$\tau_{xz} = \frac{\partial \phi}{\partial y}$ and $\tau_{yz} = -\frac{\partial \phi}{\partial x}$ where ϕ is Prandtl's stress function

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = G\beta\left(-y + \frac{\partial \psi}{\partial x}\right) \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x} = G\beta\left(x + \frac{\partial \psi}{\partial y}\right)$$

Compatibility (i.e., displacement continuity): $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$

$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = -\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 2G\beta + \underbrace{\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial^2 \psi}{\partial x \partial y}}_{=0 \text{ for continuity to hold}} = 2G\beta \qquad \therefore \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\beta$$

Now, we re-write the BC in terms of ϕ:

$$T_n = S\hat{n} = \tau_{xz} n_x + \tau_{yz} n_y = \frac{\partial \phi}{\partial y} n_x - \frac{\partial \phi}{\partial x} n_y = 0$$

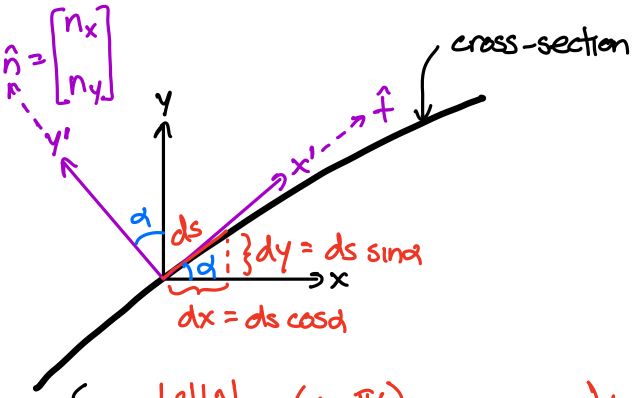
Consider the variation of Prandtl's stress function on boundary, $\phi(s) = \phi[x(s), y(s)]$, then:

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial \phi}{\partial x} n_y - \frac{\partial \phi}{\partial y} n_x = 0 \leftarrow$$

$$\frac{\partial \phi}{\partial s} = 0 \quad \therefore \quad \phi = \text{const.}$$

Since no constant is better then another, let $\phi = 0$.



$$\begin{cases} n_x = |\hat{i}||\hat{n}|\cos(\alpha + \pi/2) = -\sin\alpha = -\frac{dy}{ds} \\ n_y = |\hat{j}||\hat{n}|\cos(\alpha) = \frac{dx}{ds} \end{cases}$$

$T = 2\int \phi \, dA$ applied torque

Use Prandtl's stress function to determine the warping function of a prismatic bar with circular cross-section. Plot the corresponding stress field.

$$\phi = K\left(\frac{x^2}{r^2} + \frac{y^2}{r^2} - 1\right)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = K\frac{4}{r^2} = -2G\beta \qquad \therefore \qquad K = -\frac{G\beta r^2}{2}$$

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -G\beta y = G\beta\left(-y + \frac{\partial\psi}{\partial x}\right) \qquad \tau_{yz} = -\frac{\partial\phi}{\partial x} = G\beta x = G\beta\left(x + \frac{\partial\psi}{\partial y}\right)$$

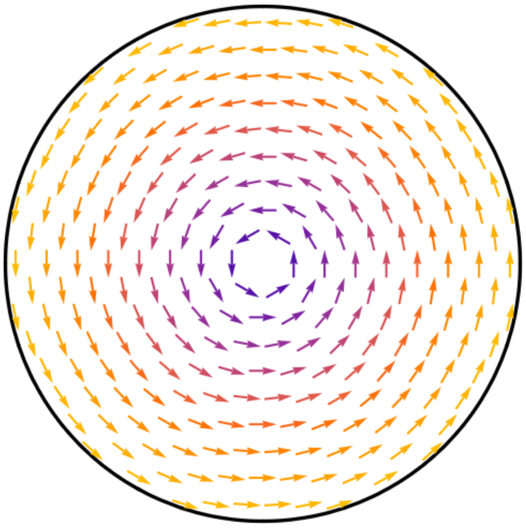
$$\frac{\partial\psi}{\partial x} = 0 \quad \therefore \quad \psi = \int\frac{\partial\psi}{\partial x}\,dx = f(y) \qquad \frac{\partial\psi}{\partial y} = 0 \quad \therefore \quad \psi = \int\frac{\partial\psi}{\partial y}\,dy = f(x)$$

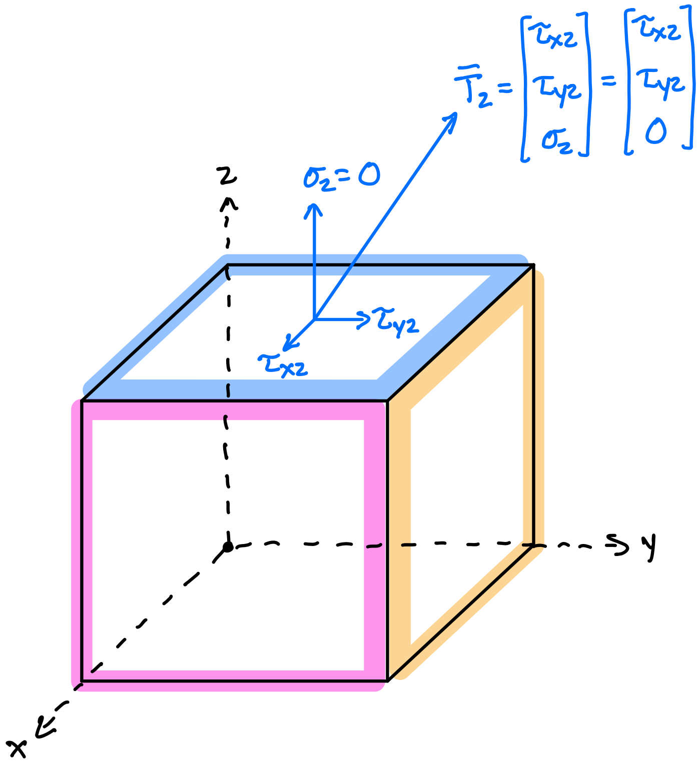
$$\psi(x,y) = f(y) = f(x) = \text{const.}$$

$$\tau_{xz} = -G\beta y \qquad \tau_{yz} = G\beta x$$

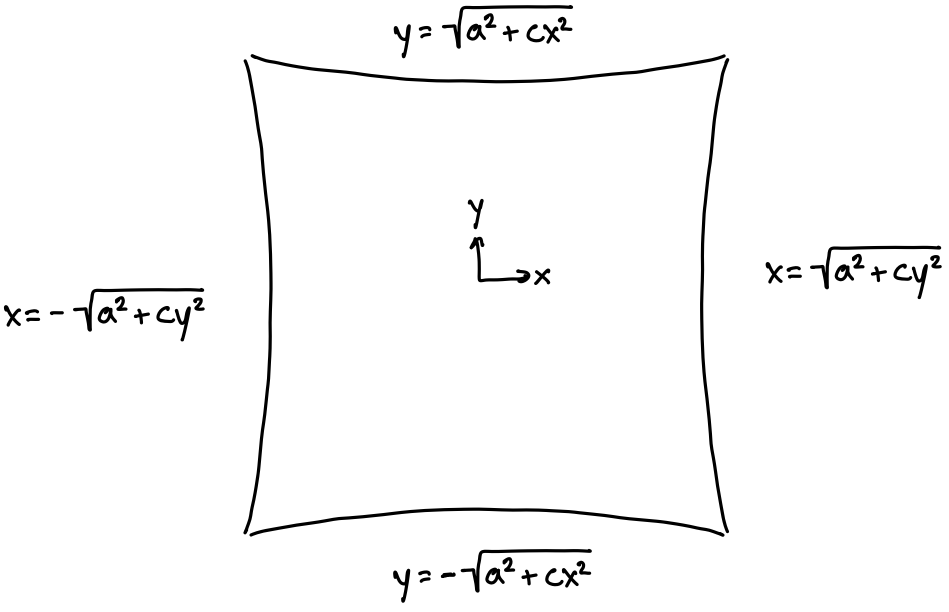
$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} \qquad \hat{n}_\tau = \frac{1}{\tau}\begin{bmatrix}\tau_{xz} \\ \tau_{yz}\end{bmatrix}$$

Recall: τ_{xz} and τ_{yz} are components of the stress vector stress traction.





Use Prandtl's stress function to determine the warping function of a prismatic bar with convex square cross-section. Plot the corresponding stress field.



$x + \sqrt{a^2 + cy^2} = 0$ left $\qquad y + \sqrt{a^2 + cx^2} = 0$ bottom

$x - \sqrt{a^2 + cy^2} = 0$ right $\qquad y - \sqrt{a^2 + cx^2} = 0$ top

$$f(x,y) = (x + \sqrt{a^2 + cy^2})(x - \sqrt{a^2 + cy^2})(y + \sqrt{a^2 + cx^2})(y - \sqrt{a^2 + cx^2})$$

$$= (a^2 + cx^2 - y^2)(a^2 - x^2 + cy^2)$$

$$\phi = \kappa f(x,y) = \kappa(a^2 + cx^2 - y^2)(a^2 - x^2 + cy^2)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4a^2\kappa(c-1) + 2\kappa[1 + c(c-6)](x^2 + y^2) = -2G\beta$$

$1 + c(c-6) = 0 \quad \therefore \quad c = 3 \pm 2\sqrt{2}$

Prandtl's method only works (analytically) for certain cross-sections; otherwise, numerically solve $\nabla^2\psi = 0$.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4a^2\kappa(c-1) = -2G\beta \quad \therefore \quad \kappa = -\frac{G\beta}{2a^2(c-1)}$$

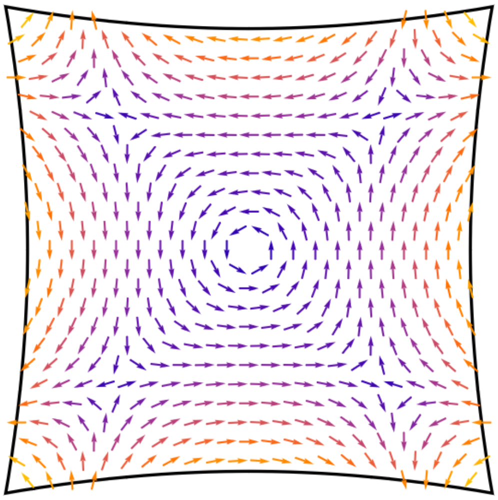
$$\tau_{xz} = \frac{\partial\phi}{\partial y} = 2\kappa y\left[(1+c^2)x^2 - 2cy^2 - a^2(1-c)\right] = G\beta\left(-y + \frac{\partial\psi}{\partial x}\right)$$

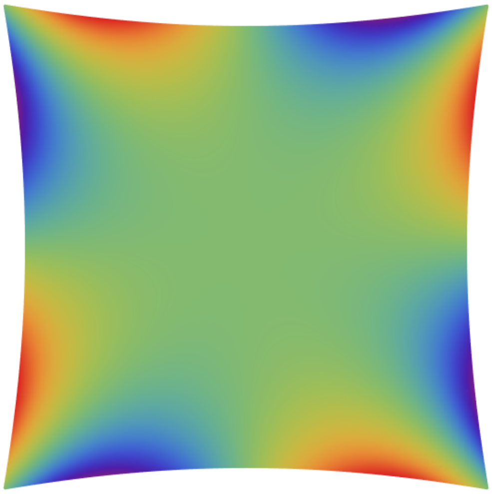
$$\frac{\partial\psi}{\partial x} = \frac{2\kappa y\left[(1+c^2)x^2 - 2cy^2 - a^2(1-c)\right]}{G\beta} + y \qquad \therefore \qquad \psi(x,y) = xy + \frac{2\kappa xy\left[(1+c^2)x^2 - 6cy^2 - 3a^2(1-c)\right]}{3G\beta} + f(y)$$

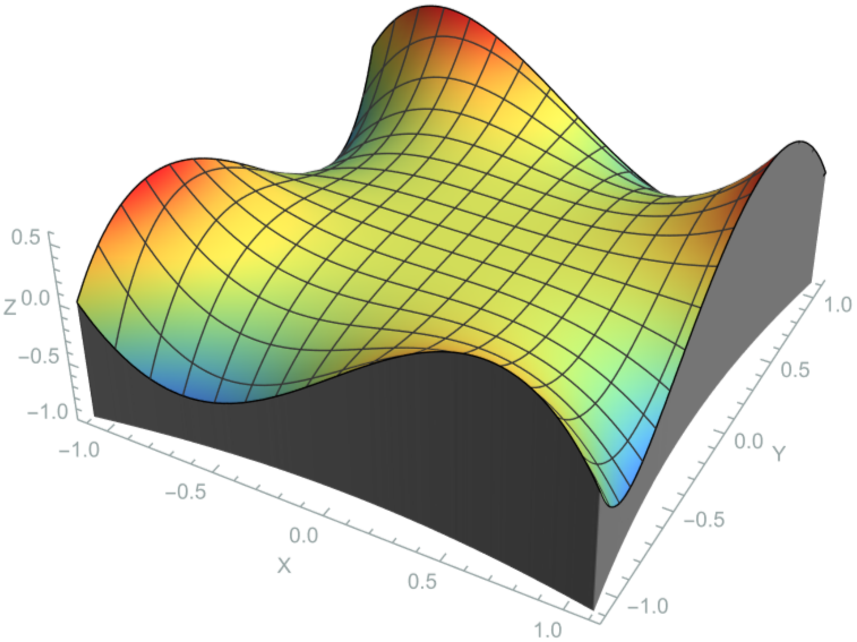
$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = -2\kappa x\left[(1+c^2)y^2 - 2cx^2 - a^2(1-c)\right] = G\beta\left(x + \frac{\partial\psi}{\partial y}\right)$$

$$\frac{\partial\psi}{\partial y} = -\frac{2\kappa x\left[(1+c^2)y^2 - 2cx^2 - a^2(1-c)\right]}{G\beta} - x \qquad \therefore \qquad \psi(x,y) = -xy - \frac{2\kappa xy\left[(1+c^2)y^2 - 6cx^2 - 3a^2(1-c)\right]}{3G\beta} + f(x)$$

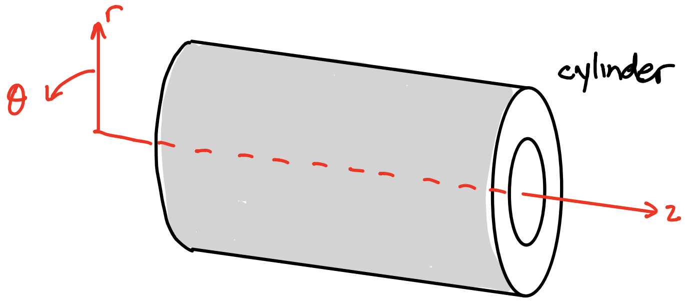
$$\psi(x,y) = \frac{\kappa}{3G\beta}\left[1 + c(6+c)\right](x^2 - y^2)xy$$



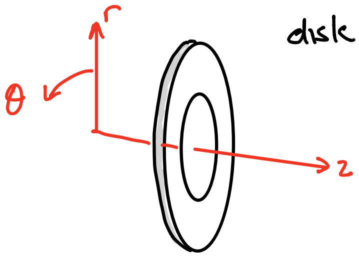




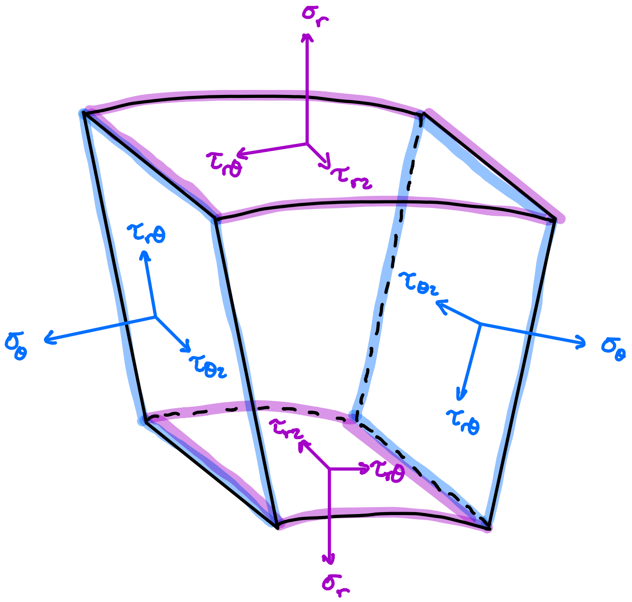
TOPIC 7:
Problems in Elasticity, Part III: Axisymmetric Problems

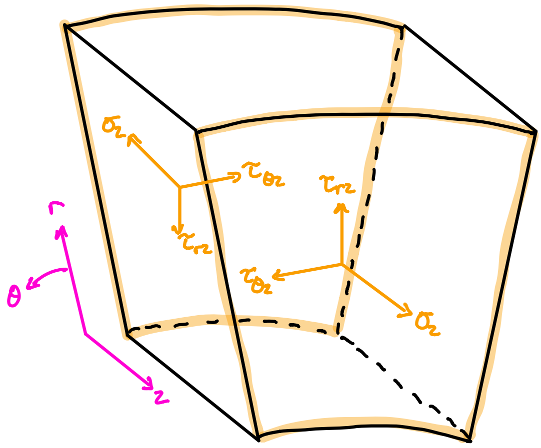


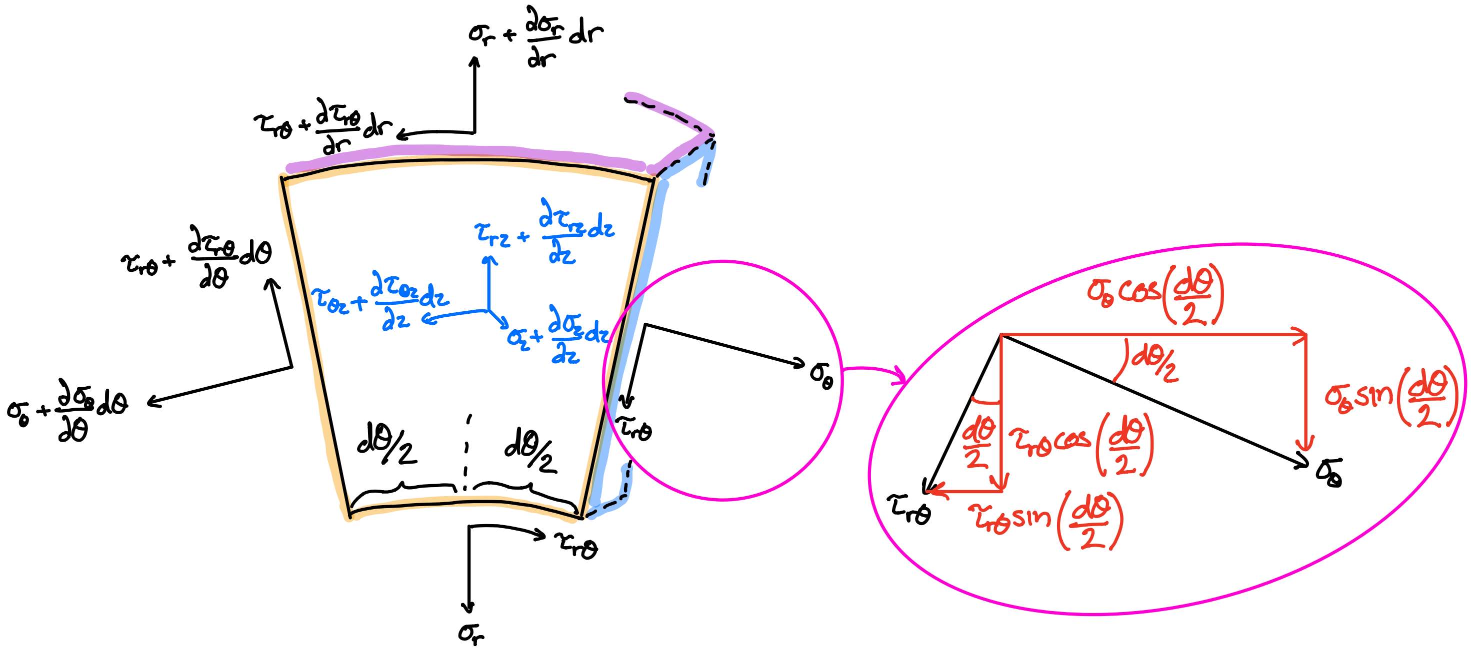
cylinder (plane strain)



disk (plane stress)







$$\sum F_r = -\sigma_r r d\theta dz + \left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right)(r+dr) d\theta dz - \tau_{rz} r d\theta dr + \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial z} dz\right) r d\theta dr$$

$$- \sigma_\theta \sin\left(\frac{d\theta}{2}\right) dr dz - \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right) \sin\left(\frac{d\theta}{2}\right) dr dz - \tau_{r\theta} \cos\left(\frac{d\theta}{2}\right) dr dz + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d\theta\right) \cos\left(\frac{d\theta}{2}\right) dr dz$$

$$= \sigma_r dr d\theta dz + r \frac{\partial \sigma_r}{\partial r} dr d\theta dz + \cancelto{\approx 0}{\frac{\partial \sigma_r}{\partial r} dr^2 d\theta dz} + r \frac{\partial \tau_{rz}}{\partial z} dr d\theta dz - \sigma_\theta dr d\theta dz - \cancelto{\approx 0}{\frac{1}{2} \frac{\partial \sigma_\theta}{\partial \theta} dr d\theta^2 dz} + \frac{\partial \tau_{r\theta}}{\partial \theta} dr d\theta dz = 0$$

$$= \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\sigma_r - \sigma_\theta) + b_r = 0 \qquad b_r = \rho \omega^2 r$$

$$\Sigma F_\theta = \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + b_\theta = 0$$

$$\Sigma F_z = \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + b_z = 0 \qquad b_z = \rho g$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \rho\omega^2 r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + b_\theta = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + \rho g = 0$$

plane stress: one side much smaller than other two; stress confined to xy-plane. (disk)

plane strain: one side much longer than other two; stress constant along the length; negligible strain in z-direction. (pipe)

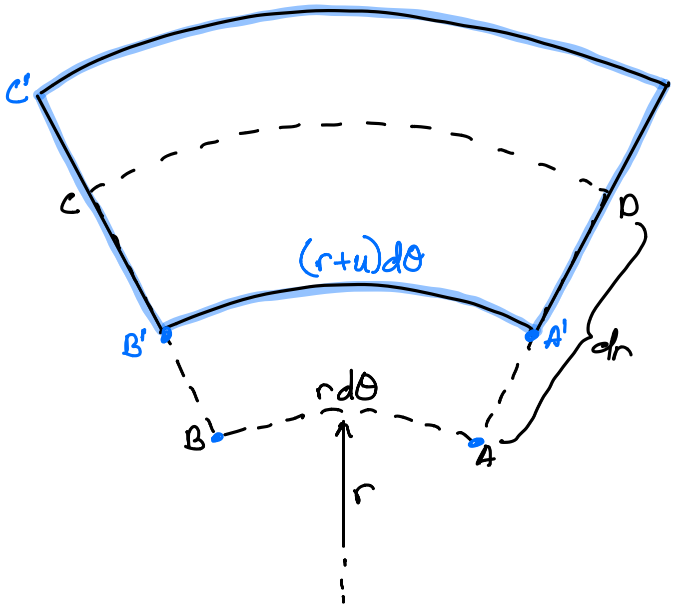
∴ For plane stress/strain conditions, $\partial/\partial z = 0$ and $\sigma_z = \tau_{rz} = \tau_{\theta z} = 0$.

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \rho\omega^2 r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\tau_{r\theta} + b_\theta = 0$$

when loading is independent of θ and there are no torsional loads.

$$\boxed{\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \rho\omega^2 r = 0}$$



u: radial displacements

v: azimuthal (θ) displacements

$$u_D = u_A + \frac{du}{dr}dr$$

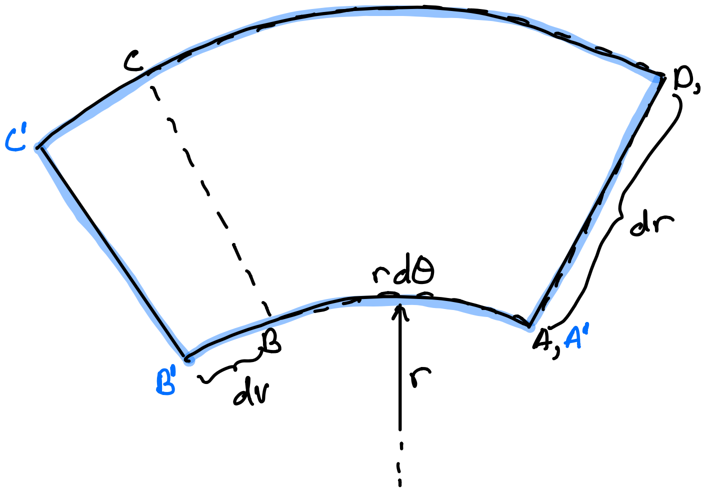
$$du = u_D - u_A = \frac{du}{dr}dr$$ change in length in radial direction

$$\varepsilon_r = \frac{du}{dr}$$ divided by original length, dr

$$v_B = v_A + u\,d\theta$$

$$dv = v_B - v_A = u\,d\theta$$ change in length in azimuthal direction due to radial displacement, u.

$$\varepsilon_\theta = \frac{u\,d\theta}{r\,d\theta} = \frac{u}{r}$$ divided by original length, rdθ.



$$v_B = v_A + \frac{\partial v}{\partial \theta} d\theta$$

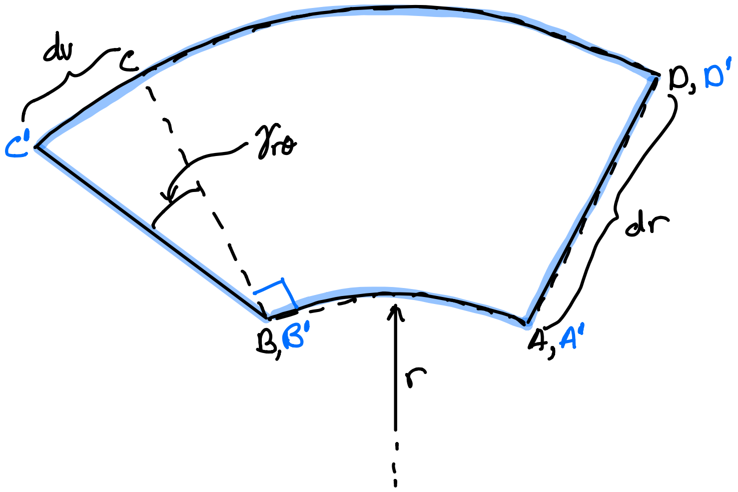
$$dv = v_B - v_A = \frac{\partial v}{\partial \theta} d\theta$$

change in length in azimuthal direction due to azimuthal displacement, v.

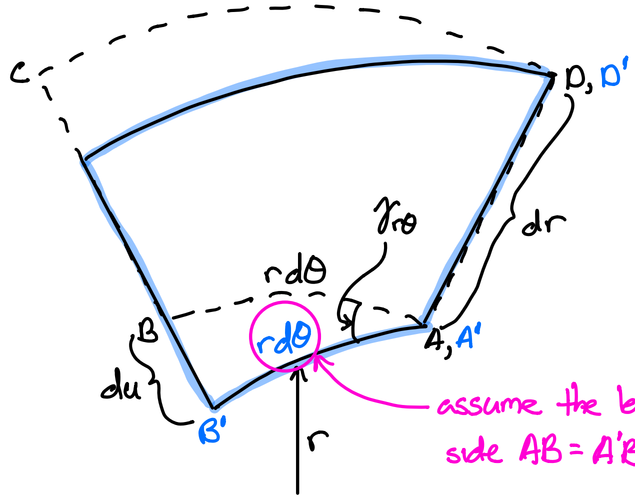
$$\varepsilon_\theta = \frac{\frac{\partial v}{\partial \theta} d\theta}{r\,d\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

divided by original length, rdθ.

$$\therefore \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$



$$\tan\gamma_{r\theta} \approx \gamma_{r\theta} = \frac{\partial v}{dr}$$



assume the length of side AB = A'B'

$$\tan\gamma_{r\theta} \approx \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \boxed{\frac{v}{r}}$$

removes the contribution due to stretching in v-direction that does not result in angle change.

$$\left.\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) \\ \sigma_\theta &= \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \\ \tau_{r\theta} &= G\gamma_{r\theta} \end{aligned}\right\} \text{plane stress}$$

$$\left.\begin{aligned} \sigma_r &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_r + \nu\varepsilon_\theta] \\ \sigma_\theta &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_\theta + \nu\varepsilon_r] \\ \tau_{r\theta} &= G\gamma_{r\theta} \end{aligned}\right\} \text{plane strain}$$

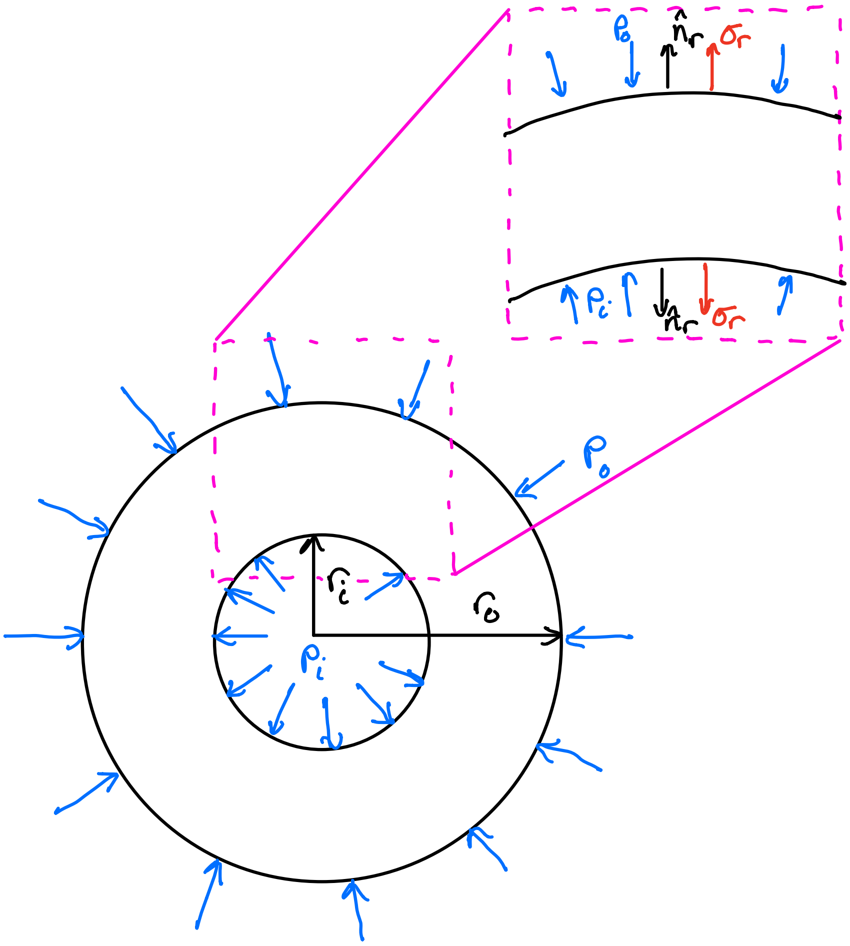
For axisymmetric problems, all $\frac{\partial}{\partial\theta} \to 0$; thus $\varepsilon_\theta = \frac{u}{r}$ and $\gamma_{r\theta} = -\frac{v}{r}$.

Substitute $\varepsilon_r = \frac{du}{dr}$ and $\varepsilon_\theta = \frac{u}{r}$ into constitutive equations for plane stress/strain, then substitute σ_r and σ_θ into equilibrium equation: $\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \rho\omega^2 r = 0$.

Plane stress route:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = -\frac{1-\nu^2}{E}\rho\omega^2 r$$

$$\left.\begin{aligned}
u(r) &= c_1 r + \frac{c_2}{r} - \frac{1-\nu^2}{8E}\rho\omega^2 r^3 \\
\sigma_r(r) &= \frac{E}{1-\nu^2}\left[(1+\nu)c_1 - \frac{1-\nu}{r^2}c_2\right] - \frac{3+\nu}{8}\rho\omega^2 r^2 \\
\sigma_\theta(r) &= \frac{E}{1-\nu^2}\left[(1+\nu)c_1 + \frac{1-\nu}{r^2}c_2\right] - \frac{1-3\nu}{8}\rho\omega^2 r^2
\end{aligned}\right\} c_1 \text{ and } c_2 \text{ determined from boundary conditions}$$



σ_r is positive when it points in the same direction as the surface normal $\hat{n}$.

p_o and p_i act against the surface normal, so they act as a negative stress

Boundary Conditions (assuming $\omega = 0$)

1. Solid disk ($r_i = 0$): $c_2 = 0$ to avoid infinites.

$$\sigma_r(r = r_o) = \frac{E}{1-\nu} c_1 = -p_o$$

$$\therefore \; c_1 = -\frac{(1-\nu)p_o}{E}$$

$$u(r) = c_1 r = -\frac{(1-\nu)p_o}{E} r$$

$$\sigma_r(r) = \frac{E}{1-\nu} c_1 = -p_o$$

$$\sigma_\theta(r) = \frac{E}{1-\nu} c_1 = -p_o$$

2. Holey disk ($r_i \neq 0$):

$$\sigma_r(r = r_o) = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 - \frac{1-\nu}{r_o^2} c_2\right] = -p_o$$

$$\sigma_r(r = r_i) = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 - \frac{1-\nu}{r_i^2} c_2\right] = -p_i$$

$$\therefore \; c_1 = \frac{1-\nu}{E}\left(\frac{r_i^2 p_i - r_o^2 p_o^2}{r_o^2 - r_i^2}\right)$$

$$c_2 = \frac{1+\nu}{E}\left[\frac{r_i^2 r_o^2 (p_i - p_o)}{r_o^2 - r_i^2}\right]$$

$$u(r) = c_1 r + \frac{c_2}{r} = \frac{1}{E(r_o^2 - r_i^2)}\left[r(1-\nu)(p_i r_i^2 - p_o r_o^2) + \frac{r_i^2 r_o^2}{r}(1+\nu)(p_i - p_o)\right]$$

$$\sigma_r(r) = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 - \frac{1-\nu}{r^2} c_2\right] = \frac{1}{r_o^2 - r_i^2}\left[r_i^2 p_i\left(1 - \frac{r_o^2}{r^2}\right) - r_o^2 p_o\left(1 - \frac{r_i^2}{r^2}\right)\right]$$

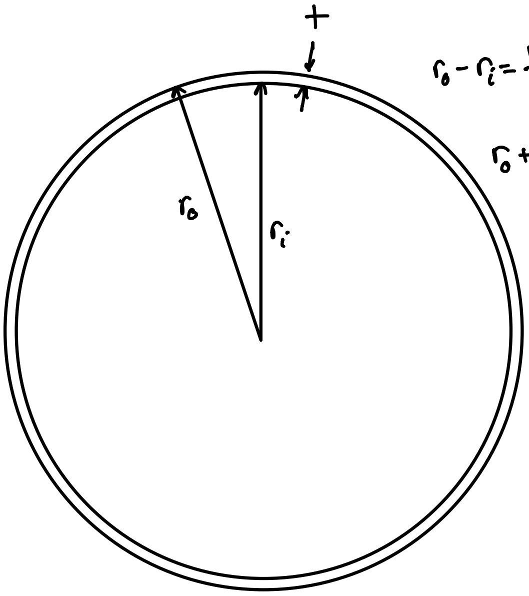
$$\sigma_\theta(r) = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 + \frac{1-\nu}{r^2} c_2\right] = \frac{1}{r_o^2 - r_i^2}\left[r_i^2 p_i\left(1 + \frac{r_o^2}{r^2}\right) - r_o^2 p_o\left(1 + \frac{r_i^2}{r^2}\right)\right]$$

3. Holey disk with $p_i = p_o = p$:

$$u(r) = -\frac{1-\nu}{E} p r$$

$$\sigma_r(r) = -p$$

$$\sigma_\theta(r) = -p$$



$r_o - r_i = t$

$r_o + r_i \approx 2r_o \approx 2r_i = 2r$ $r_i \approx r_o = r$

$$\sigma_r = \frac{1}{r_o^2 - r_i^2}\left[r_i^2 p_i\left(1 - \frac{r_o^2}{r^2}\right) - r_o^2 p_o\left(1 - \frac{r_i^2}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{1}{\boxed{r_o^2 - r_i^2}}\left[r_i^2 p_i\left(1 + \frac{r_o^2}{r^2}\right) - r_o^2 p_o\left(1 + \frac{r_i^2}{r^2}\right)\right]$$

$$\underbrace{(r_o - r_i)}_{t}\underbrace{(r_o + r_i)}_{2r}$$

$t \approx 0$

$r_o >> t$

$$\sigma_r = \frac{1}{2r_o t}\left((r_o - t)^2\left[p_i\left(1 - \frac{r_o^2}{r^2}\right) + p_o\frac{r_o^2}{r^2}\right] - p_o r_o^2\right)$$

$$\sigma_\theta = \frac{1}{r_o t}\left[(r_o - t)^2 p_i - p_o r_o^2\right]$$

$\sigma_r \approx 0$ thin disk

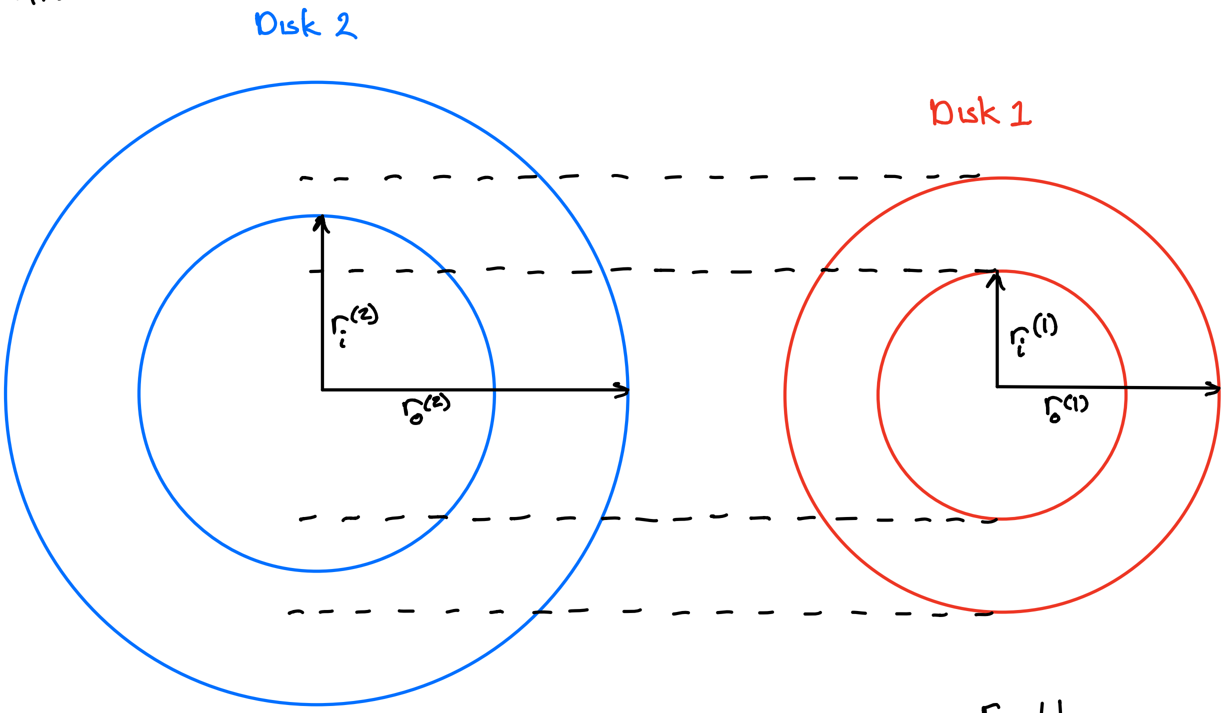
$\sigma_\theta \approx \frac{(p_i - p_o)r}{t}$ hoop stress

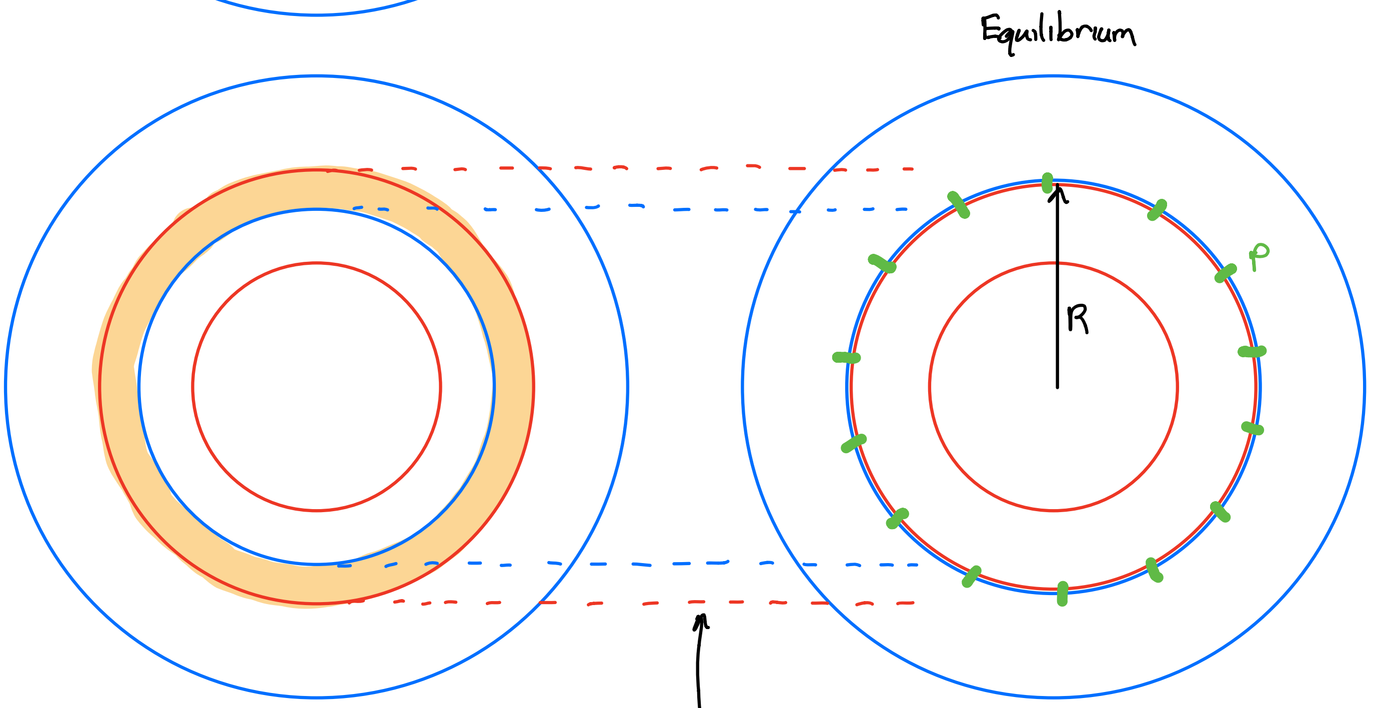
Thin Disk: $r = r_i \approx r_o$; thickness $t = r_o - r_i$

$\sigma_r \approx 0$

$$\sigma_\theta = \frac{2}{\underbrace{(r_o+r_i)}_{2r}\underbrace{(r_o-r_i)}_{t}}(r_i^2 p_i - r_o^2 p_o) = \frac{(p_i - p_o)r}{t}$$ same for thin-walled pressure vessel

Press Fit:





The outter radius of disk 1, $r_o^{(1)}$ is greater than the inner radius of disk 2, $r_i^{(2)}$.

radial interference / mismatch

$\Delta = r_o^{(1)} - r_i^{(2)}$

To fit disk 1 into disk 2, the outter radius of disk 1 must shrink to R and the inner radius of disk 2 must expand to R (final, equilibrium radius).

The press-fit creates an interface pressure, p.

$$R = r_o^{(1)} + u^{(1)}\Big|_{r=r_o^{(1)}}$$ final radius from perspective of disk 1

$$-\quad R = r_i^{(2)} + u^{(2)}\Big|_{r=r_i^{(2)}}$$ final radius from perspective of disk 2

$$0 = \underbrace{r_o^{(1)} - r_i^{(2)}}_{\Delta} + u^{(1)}\Big|_{r=r_o^{(1)}} - u^{(2)}\Big|_{r=r_i^{(2)}} \qquad \therefore \qquad u^{(2)}\Big|_{r=r_i^{(2)}} - u^{(1)}\Big|_{r=r_o^{(1)}} = \Delta$$

$$u = \frac{1}{E(r_o^2 - r_i^2)}\left[r(1-\nu)(p_i r_i^2 - p_o r_o^2) + \frac{r_i^2 r_o^2}{r}(1+\nu)(p_i - p_o)\right]$$

For disk 1: final outter radius $r_o^{(1)} = R$ and $p_o = p$ (assume $p_i = 0$)

For disk 2: final inner radius $r_i^{(2)} = R$ and $p_i = p$ (assume $p_o = 0$)

Assuming same material properties, $E = E^{(1)} = E^{(2)}$:

$$p = \frac{\Delta E}{2R^3}\,\frac{(R^2 - [r_i^{(1)}]^2)(R^2 - [r_o^{(2)}]^2)}{[r_i^{(1)}]^2 - [r_o^{(2)}]^2}$$

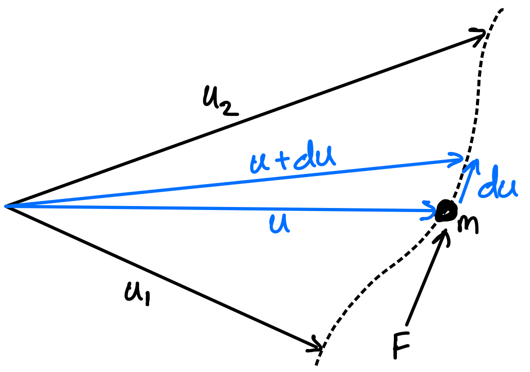
TOPIC 8:
Work and Strain Energy

kinetic energy (T): energy of motion

potential energy (V): energy of configuration

work (W): energy added/removed from a system that, necessarily, changes its kinetic and/or potential energy

Consider a particle moving under the action of a force.



incremental work: $dW = F \cdot du$

$$dW = F \cdot du = m\ddot{u} \cdot du = m\frac{d\dot{u}}{dt} \cdot \frac{du}{dt}dt = m\frac{d\dot{u}}{dt}dt \cdot \frac{du}{dt} = m\frac{du}{dt} \cdot d\dot{u} = d\left(\frac{1}{2}m\dot{u} \cdot \dot{u}\right) = dT$$

The work performed in moving a particle from u_1 to u_2 is responsible for a change in the kinetic energy

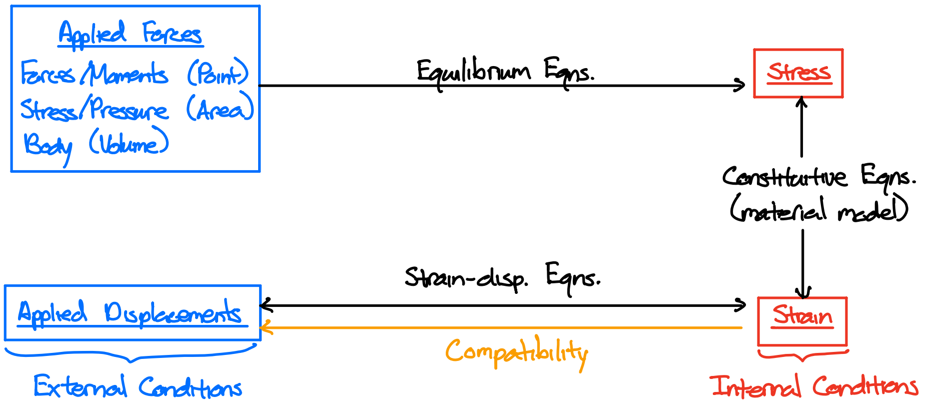
$$W = \int_{u_1}^{u_2} F \cdot du = \int_{\dot{u}_1}^{\dot{u}_2} d\left(\frac{1}{2}m\dot{u} \cdot \dot{u}\right) = \frac{1}{2}m\dot{u}_2 \cdot \dot{u}_2 - \frac{1}{2}m\dot{u}_1 \cdot \dot{u}_1 = \Delta T$$

The work performed in moving a particle from u_1 to u_2 under the influence of a conservative force, F_c, is independent of the path taken.

Consider the work done by a conservative force as the particle moves from u_1 to u_2 through a referene point, u_r:

$$W = \int_{u_1}^{u_2} F_c \cdot du = \int_{u_1}^{u_r} F_c \cdot du + \int_{u_r}^{u_2} F_c \cdot du = \underbrace{\int_{u_1}^{u_r} F_c \cdot du}_{V_1} - \underbrace{\int_{u_2}^{u_r} F_c \cdot du}_{V_2} = -\Delta V$$

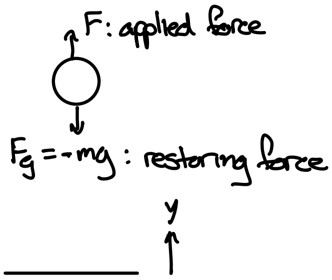
The work done by a conservative force as the particle moves from u_1 to u_2 is equal to the negative of the change in potential energy.



Energy: the ability to do work; alter the system

kinetic energy (T): energy of motion (dynamics)

potential energy: energy of configuration (displacement/deformation)



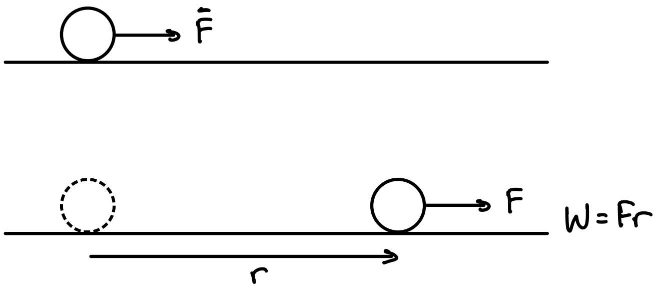
$U = mgy$

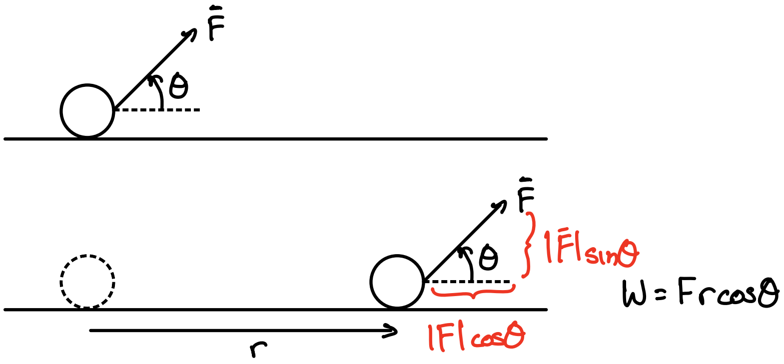
$F_g = -\frac{\partial U}{\partial y} = -mg$

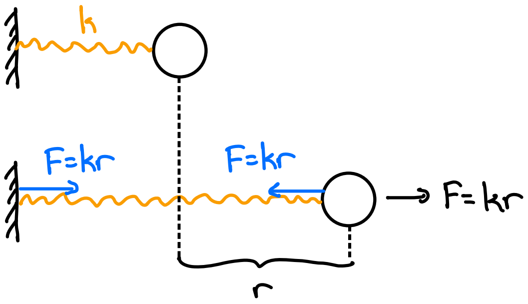
$W = mg\int dy = mgy = U$

$d W = \bar{F} \cdot d\bar{r}$ $\quad W = \int \bar{F} \cdot d\bar{r}$ $\quad W = \int \bar{M} \cdot d\bar{\theta}$

work: product of a load and the corresponding displacement that it causes

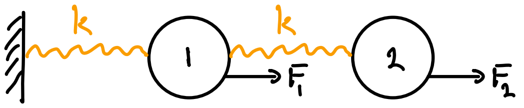






$$W = \int \bar{F} \cdot d\bar{r} = \int kr\,dr = \frac{1}{2}kr^2 = \underbrace{\frac{1}{2}(kr)r = \frac{1}{2}Fr}_{\text{if F is a linear function of displacement}} = U$$

$$F = -\frac{\partial U}{\partial r} = -kr$$

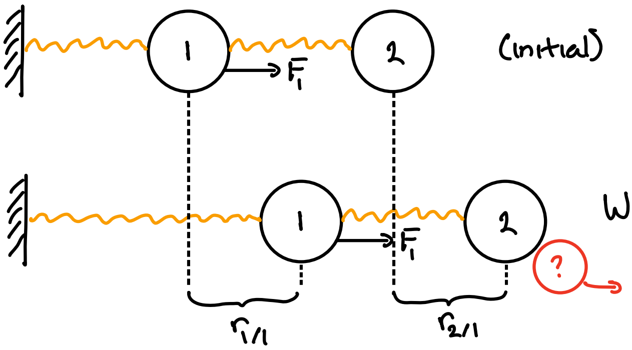


Apply F_1 first:

$r_{x/y}$ (displacement at point X due to force Y)

↑ force

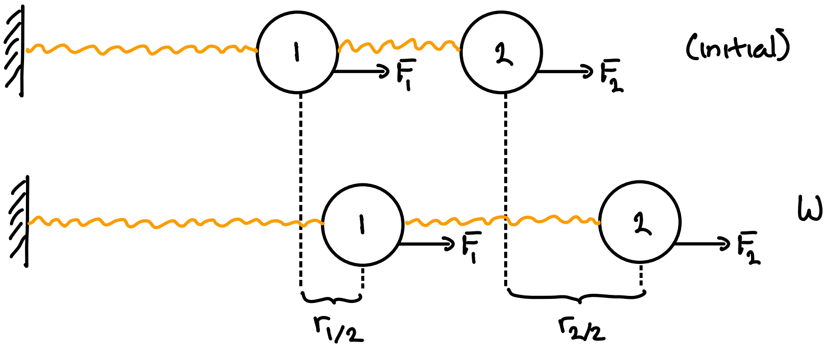
location



$W_1 = \frac{1}{2} F_1 r_{1/1}$ (work at step 1)

→ displacement, $r_{2/1}$, but no force, so no work

Now apply F_2:

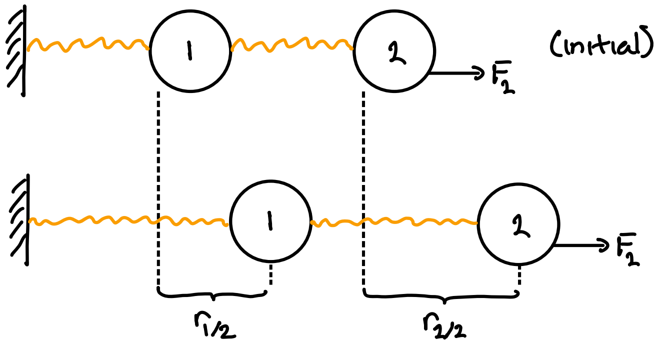


$W_2 = \frac{1}{2} F_2 r_{2/2} + \boxed{F_1 r_{1/2}}$ (work at step 2)

→ F_1 is still applied while $r_{1/2}$

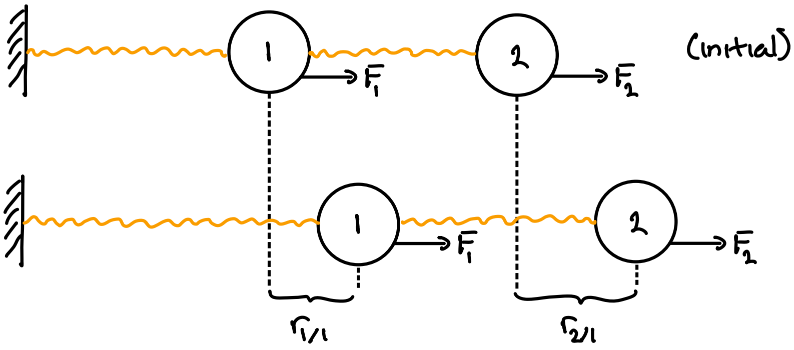
$W = W_1 + W_2 = \frac{1}{2} F_1 r_{1/1} + \frac{1}{2} F_2 r_{2/2} + F_1 r_{1/2}$

Apply F_2 first:



$W_1 = \frac{1}{2} F_2 r_{2/2}$ (work at step 1)

Now apply F_1:



$$W_2 = \frac{1}{2} F_1 r_{1/1} + F_2 r_{2/1}$$

$$W = W_1 + W_2 = \frac{1}{2} F_2 r_{2/2} + \frac{1}{2} F_1 r_{1/1} + F_2 r_{2/1}$$

The order in which the forces are applied is irrelevant.

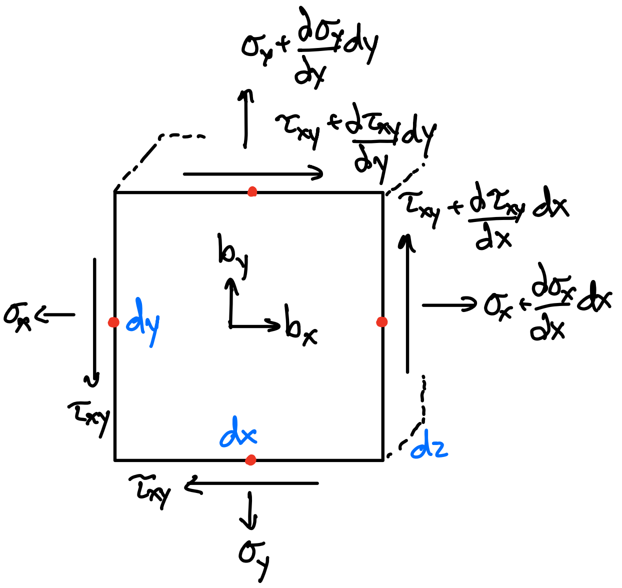
$$\frac{1}{2} F_1 r_{1/1} + \frac{1}{2} F_2 r_{2/2} + F_1 r_{1/2} = \frac{1}{2} F_2 r_{2/2} + \frac{1}{2} F_1 r_{1/1} + F_2 r_{2/1}$$

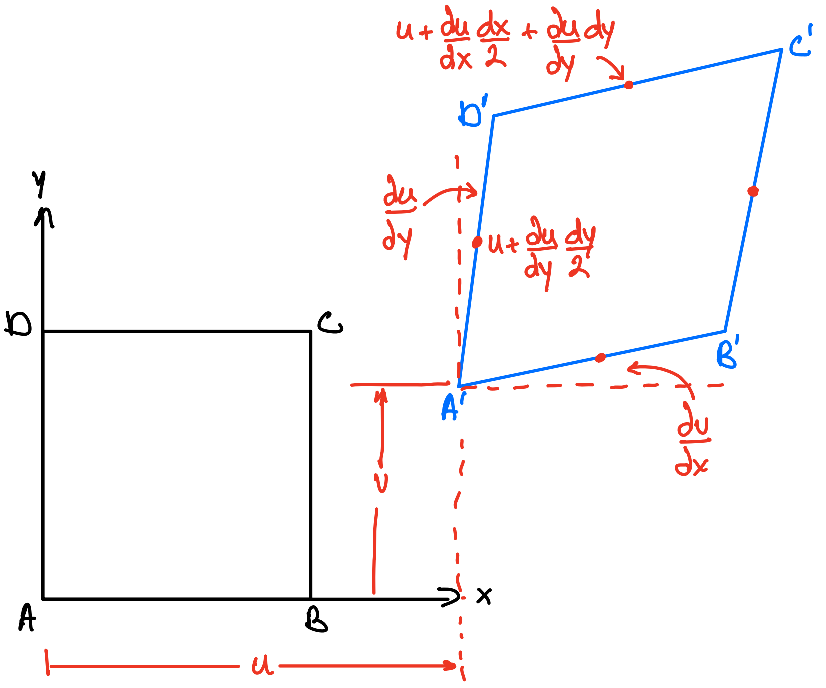
$F_1 r_{1/2} = F_2 r_{2/1}$

$r_{1/2} = r_{2/1}$

Maxwell's Reciprocity Theorem: the work done by one set of forces owing to displacements due to a second set is equal to the work done by the second set of forces owing to displacements due to the first.

Work Done on Deformable Body





$$dW = -(\sigma_x dy dz)\, d\left(u + \frac{\partial u}{\partial y}\frac{dy}{2}\right) + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right) d\left(u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}\frac{dy}{2}\right) dy dz$$

$$- (\tau_{xy} dx dz)\, d\left(u + \frac{\partial u}{\partial x}\frac{dx}{2}\right) + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}dy\right)(dx dz)\, d\left(u + \frac{\partial u}{\partial x}\frac{dx}{2} + \frac{\partial u}{\partial y}dy\right)$$

$$+ b_x (dx dy dz)\, d\left(u + \frac{\partial u}{\partial x}\frac{dx}{2} + \frac{\partial u}{\partial y}\frac{dy}{2}\right) \qquad \text{x-direction}$$

Note: $d\left(u + \frac{\partial u}{\partial x}dx\right) = du + d\varepsilon_x dx + \varepsilon_x \cancelto{0}{(d\,dx)}$ since dx is constant

$$dW = \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x\right) du + \left(\frac{\partial \sigma_x}{\partial x} + \frac{1}{2}\frac{\partial \tau_{xy}}{\partial y} + \frac{1}{2}b_x\right) d\varepsilon_x dx + \left(\frac{1}{2}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{1}{2}b_x\right) d\left(\frac{\partial u}{\partial y}\right) dy\right.$$

$$\left. + \sigma_x d\varepsilon_x + \tau_{xy}\, d\left(\frac{\partial u}{\partial y}\right)\right] dx dy dz \qquad \text{x-direction}$$

This is the equilibrium equation in the x-direction. For equilibrium to be maintained, it must be zero.

Differentials represent very small changes in quantity. The product of multiple differentials is quite small, e.g., $dx dx \ll dx$. Above $d\varepsilon_x dx$ and $d\left(\frac{\partial u}{\partial y}\right) dy$ are much much smaller than $d\varepsilon_x$ and $d\left(\frac{\partial u}{\partial y}\right)$, so much so that they can be ignored (≈ 0) since they do not have a big impact on the final result.

$$dW = \left[\sigma_x d\varepsilon_x + \tau_{xy}\, d\left(\frac{\partial u}{\partial y}\right)\right] \underbrace{dx dy dz}_{dV:\ \text{differential volume}} \qquad \text{x-direction} \quad (1D)$$

Include work from y-direction forces/displacements:

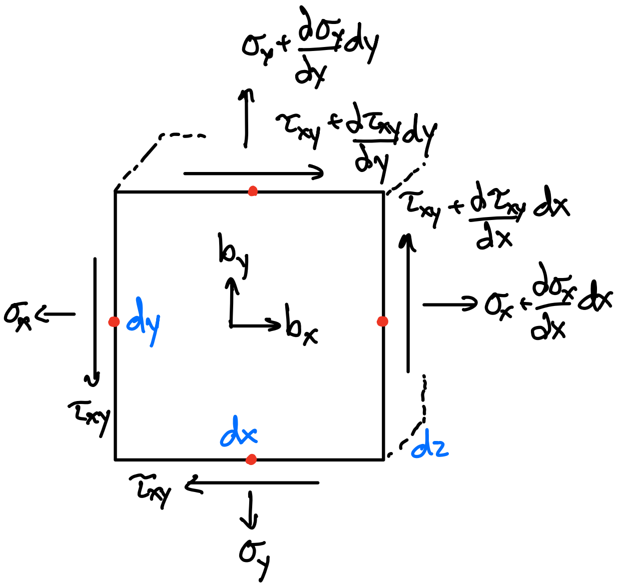
$$dW = \left[\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \tau_{xy}\, d\left(\frac{\partial u}{\partial y}\right) + \tau_{xy}\, d\left(\frac{\partial v}{\partial x}\right)\right] dV = \left[\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \tau_{xy}\, d\gamma_{xy}\right] dV \qquad (2D)$$

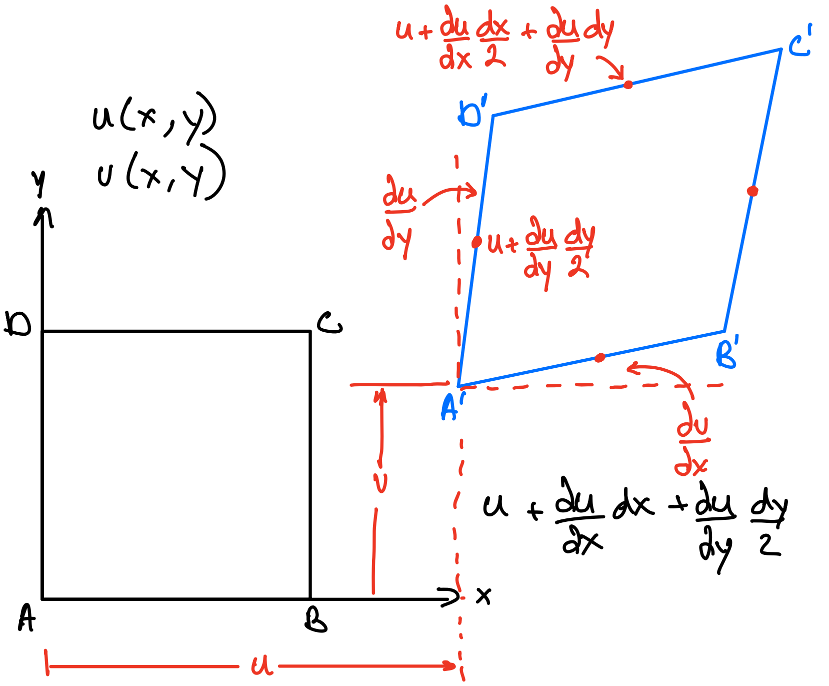
Include work from z-direction forces/moments:

$$dW = \left[\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{xz} d\gamma_{xz} + \tau_{yz} d\gamma_{yz}\right] dV \qquad (3D)$$

Integrating strains gives the total work done. In the following, we consider $d\tilde{w} = dW/dV$, the (differential) work per unit volume. We perform the analysis for 2D plane stress/strain system, and then expand the results to the 3D case.

$$dW = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right) dV$$





$$dW = F \cdot dr = -(\sigma_x\, dy dz)\, d\left(u + \frac{du}{dy}\frac{dy}{2}\right) + \left(\sigma_x + \frac{d\sigma_x}{dx} dx\right) d\left(u + \frac{du}{dx} dx + \frac{du}{dy}\frac{dy}{2}\right) dy dz - (\tau_{xy}\, dx dz)\, d\left(u + \frac{du}{dx}\frac{dx}{2}\right)$$

$$+ \left(\tau_{xy} + \frac{d\tau_{xy}}{dy} dy\right) dx dz\, d\left(u + \frac{du}{dx}\frac{dx}{2} + \frac{du}{dy} dy\right) + b_x (dx dy dz)\, d\left(u + \frac{du}{dx}\frac{dx}{2} + \frac{du}{dy}\frac{dy}{2}\right)$$

$$d\left(u + \frac{du}{dx} dx\right) = du + d\varepsilon_x dx + \cancelto{0}{\varepsilon_x (d\, dx)}$$

$$dW = \Bigg[\overset{=0}{\cancel{\left(\frac{d\sigma_x}{dx} + \frac{d\tau_{xy}}{dy} + b_x\right)}} du + \cancelto{\approx 0}{\left(\frac{d\sigma_x}{dx} + \frac{1}{2}\frac{d\tau_{xy}}{dy} + \frac{1}{2} b_x\right) d\varepsilon_x dx} + \cancelto{\approx 0}{\left(\frac{1}{2}\frac{d\sigma_x}{dx} + \frac{d\tau_{xy}}{dy} + \frac{1}{2} b_x\right) d\left(\frac{du}{dy}\right) dy}$$

$$+ \sigma_x\, d\varepsilon_x + \tau_{xy}\, d\left(\frac{du}{dy}\right)\Bigg] dx dy dz$$

$$dW = \left[\sigma_x\, d\varepsilon_x + \tau_{xy}\, d\left(\frac{du}{dy}\right)\right] dV$$ x -direction

$$dW = \left[\sigma_x\, d\varepsilon_x + \sigma_y\, d\varepsilon_y + \tau_{xy}\, d\underbrace{\left(\frac{du}{dy} + \frac{dv}{dx}\right)}_{\gamma_{xy}}\right] dV$$ x-, y- directions

$$\int \sigma_x\, d\varepsilon_x$$

$$\int E\varepsilon_x\, d\varepsilon_x$$

$$\frac{1}{2} E\varepsilon_x^2 = \frac{1}{2}\sigma_x \varepsilon_x$$

$$dW = [\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{xz} d\gamma_{xz}] dV$$ <u>xyz - direction</u>

$$dW = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}) dV$$

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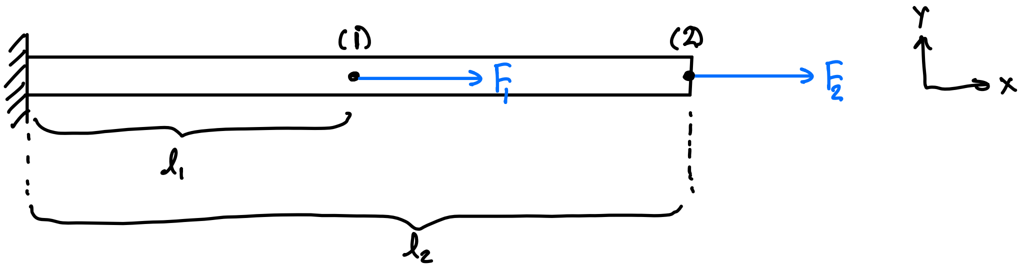
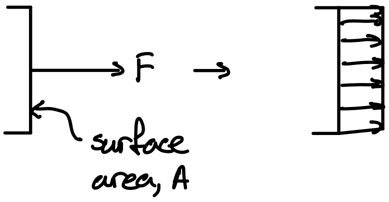
Determine work done by two (constant) forces on an elastic beam.



$$dW = \frac{1}{2}(\sigma_x \varepsilon_x + \cancelto{0}{\sigma_y \varepsilon_y} + \cancelto{0}{\sigma_z \varepsilon_z} + \cancelto{0}{\tau_{xy}\gamma_{xy}} + \cancelto{0}{\tau_{xz}\gamma_{xz}} + \cancelto{0}{\tau_{yz}\gamma_{yz}})dV = \frac{1}{2}\sigma_x \varepsilon_x dV$$

since this is a 1D problem



$\sigma_x = \frac{F}{A}$ average stress (constant)

$$W = \frac{1}{2}\int_0^l \int_A \sigma_x \varepsilon_x \, dA dx = \frac{1}{2}\int_0^l \int_A \frac{\sigma_x^2}{E} dA dx = \frac{1}{2}\int_0^l \int_A \frac{F^2}{A^2 E} dA dx = \frac{F^2 l}{2AE}$$

$\varepsilon_x = \frac{\sigma_x}{E}$

$\sigma_x = \frac{F}{A}$

integrate over area from 0 to A_{tot}

work done by constant force in 1D

Because of the linear force-displacement relationship, $W = \frac{F^2 l}{2AE} = \frac{1}{2}F\underbrace{\left(\frac{Fl}{AE}\right)}_{u}$

u: displacement at l

Let's apply F_1 first, then F_2:

$$W_1 = \frac{F_1^2 l_1}{2AE}$$

Now apply F_2:

$$W_2 = \frac{F_2^2 l_2}{2AE}$$

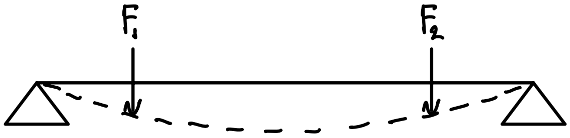
However, due to F_2, there is also displacement at (1) where F_1 is still active, and so F_1 does some work as well

$$W_{1/2} = F_1 u_{1/2} = F_1\left(\frac{F_2 l_1}{AE}\right)$$

$$W = W_1 + W_{1/2} + W_2 = \frac{F_1^2 l_1}{2AE} + F_1\left(\frac{F_2 l_1}{AE}\right) + \frac{F_2^2 l_2}{2AE}$$

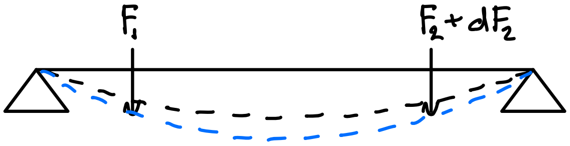
Castigliano's (second) theorem: the first partial of the strain energy w.r.t a concentrated force/moment gives the linear/angular deflection at a point.

To derive Castigliano's (second) theorem, we apply Maxwell's reciprocity theorem. Consider a simply-supported beam with two concentrated loads, F_1 and F_2, applied.



$$W_1(F_1,F_2) = U_1(F_1,F_2)$$

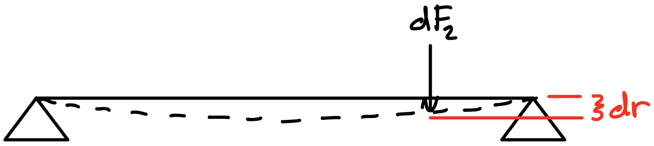
Change F_2 by an amount dF_2



$$W_2(F_1,F_2) = U_2(F_1,F_2) = U_1(F_1,F_2) + dU = U_1(F_1,F_2) + \frac{\partial U}{\partial F_2} dF_2$$

$$dU = \cancelto{0}{\frac{\partial U}{\partial F_1} dF_1} + \frac{\partial U}{\partial F_2} dF_2$$ since only F_2 changed, $dF_1 = 0$

Now we apply the differential load, dF_2, first

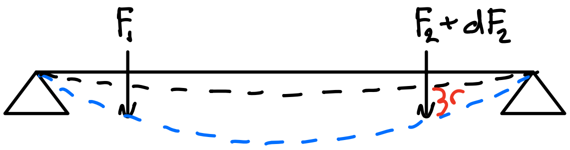


$$W_1 = \frac{1}{2} dF_2 dr$$

Add the loads F_1 and F_2 which brings us back to the final loading configuration

$$W_2(F_1,F_2) = \tilde{W}(F_1,F_2) + \frac{1}{2} dF_2 dr + dF_2 r$$

$$= W_1(F_1,F_2) + \frac{1}{2} dF_2 dr + dF_2 r$$



$$\tilde{W}(F_1,F_2) \approx W_1(F_1,F_2)$$ since dF_2 and dr are so small, the work, W, doesn't add much

By Maxwell's reciprocity theorem, the order of the applied loads does not matter, so the work done is the same in either case.

$$W_1(F_1,F_2) + \frac{\partial U}{\partial F_2} dF_2 = W_2(F_1,F_2) = W_1(F_1,F_2) + \frac{1}{2} dF_2 dr + dF_2 r$$

$$\frac{\partial U}{\partial F_2} = \cancelto{\approx 0}{\frac{1}{2} dr} + r = r$$ since $dr \ll r$

Castigliano's (second) theorem: $\frac{\partial U}{\partial P_i} = r_i$ and $\frac{\partial U}{\partial M_i} = \theta_i$

1. Calculate the total strain energy due to all loads

2. Differentiate the strain energy w.r.t. a concentrated load, the result is the relevant deflection at the location (and direction) of the load.

Castigliano's (first) theorem: $\frac{\partial U}{\partial r_i} = P_i$ and $\frac{\partial U}{\partial \theta_i} = M_i$

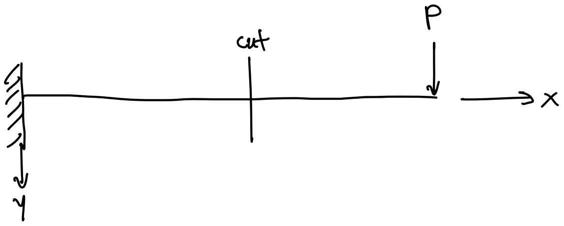
In the absence of dissipation (energy loss) the work W done in deforming an elastic body is equal to the strain energy U (the energy stored by material under deformation).

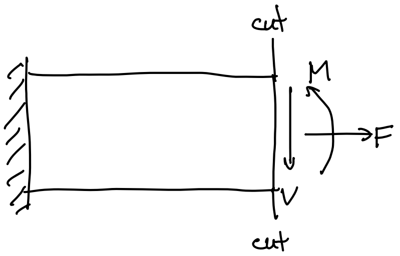
For a thin beam:

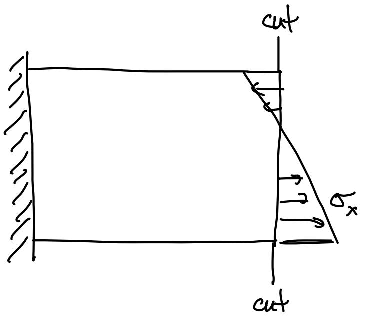
$$U = W = \frac{1}{2}\int_V (\sigma_x \varepsilon_x + \tau_{xy}\gamma_{xy})\, dV = \frac{1}{2}\int_0^\ell \int_A (\sigma_x \varepsilon_x + \tau_{xy}\gamma_{xy})\, dA\, dx$$

Recall $\varepsilon_x = \frac{\sigma_x}{E}$ and $\gamma_{xy} = \frac{\tau_{xy}}{G}$

$$U = \frac{1}{2}\int_0^\ell \int_A \left(\frac{\sigma_x^2}{E} + \frac{\tau_{xy}^2}{G}\right) dA\, dx$$







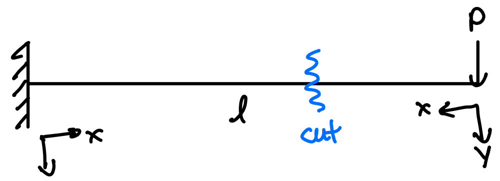
For beam bending, $\sigma_x = \frac{My}{I} + \frac{F}{A}$ and $\tau_{xy} = \frac{VQ}{Ib}$ where Q is the first moment of the cross section and b is the thickness. Thus:

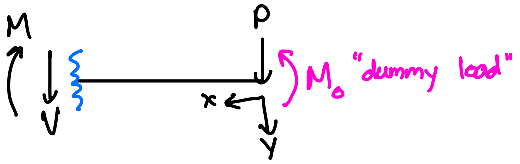
$$U = \frac{1}{2}\int_0^\ell \int_A \left(\frac{M^2}{EI^2}y^2 + \frac{2MF}{IA}y + \frac{F^2}{A^2} + \frac{V^2Q^2}{GI^2b^2}\right) dA\, dx$$

$$= \frac{1}{2}\int_0^\ell \left(\frac{M^2}{EI} + \frac{F^2}{EA} + \frac{\kappa V^2}{GA} + \frac{T^2}{GJ}\right) dx$$

where κ is form factor given by $\kappa = \frac{A}{I^2}\int \frac{Q^2}{b}\, dy$

Determine the slope at the end of the beam in the first case (ignore shear).





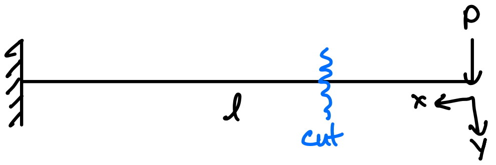
$$U = \frac{1}{2}\int_0^l \left(\frac{M^2}{EI} + \cancel{\frac{F^2}{EA}} + \cancel{\frac{kV^2}{GA}} + \cancel{\frac{T^2}{GJ}} \right) dx$$

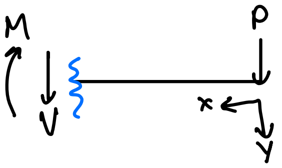
$$\Sigma M = -M - Px + M_0 = 0 \qquad \therefore \quad M = -Px + M_0$$

$$U = \frac{1}{2}\int_0^l \frac{(M_0 - Px)^2}{EI}\,dx = \frac{1}{2}\int \frac{M_0^2 - 2PM_0x + P^2x^2}{EI}\,dx = \frac{1}{2EI}\left(M_0^2 l - PM_0 l^2 + \frac{P^2 l^3}{3} \right)$$

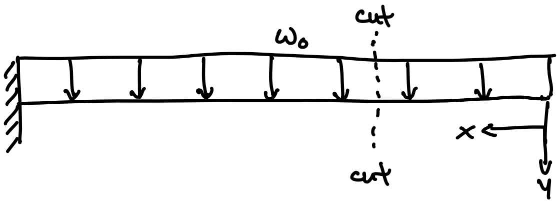
$$\theta_{tip} = \frac{\partial U}{\partial M_0} = \frac{1}{2EI}\left(2\cancelto{0}{M_0} l - Pl^2\right) = -\frac{Pl^2}{2EI}$$

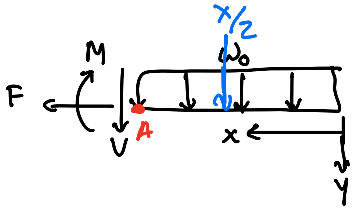
Determine the deflection at the end of a cantilever beam due to an end load P. Ignore shear.





Determine the strain energy in a cantilever with distributed load $w(x) = w_0$





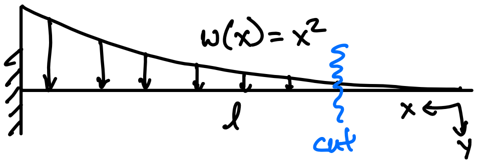
$$U = \frac{1}{2}\int_0^{l}\left(\frac{M^2}{EI} + \cancel{\frac{F^2}{EA}} + \cancel{\frac{kV^2}{GA}} + \cancel{\frac{T^2}{GJ}}\right)dx = \frac{1}{2}\int_0^{l}\frac{M^2}{EI}dx$$

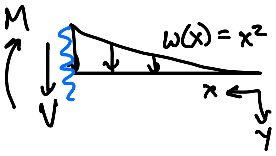
$$\underbrace{w_0 x}_{\text{force}}\ \overset{\text{moment arm}}{\underbrace{\frac{x}{2}}} = \frac{w_0 x^2}{2}$$

$\Sigma F_x = F = 0$ $\quad \Sigma F_y = V = 0$ $\quad \Sigma M_A = -M + \frac{w_0 x^2}{2} = 0$ $\quad \therefore\ M = \frac{w_0 x^2}{2}$

$$U = \frac{w_0^2}{8EI}\int_0^{l} x^4\,dx = \frac{w_0^2 l^5}{40EI}$$

Determine the strain energy in a cantilever with distributed load $w(x) = x^2$





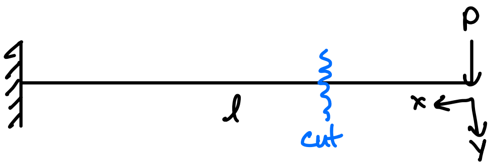
$$\Sigma M = \underbrace{[w(x)x]}_{F}\, x_c - M = \frac{3x^3}{4} - M = 0 \qquad \therefore\ M = \frac{3x^4}{4}$$

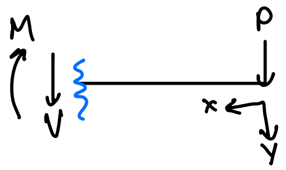
when determining the moment produced by distributed loads, the force is taken to act at the center of loading, x_c

$$x_c = \frac{\int_0^x w(s)s\,ds}{\int_0^x w(s)ds} = \frac{\int_0^x s^3 ds}{\int_0^x s^2 ds} = \frac{3x}{4}$$

$$\boxed{U = \frac{9}{32EI}\int_0^{l} x^8\,dx = \frac{l^9}{32EI}}$$

Determine the deflection at the end of a cantilever beam due to an end load P. Ignore shear.





$$\Sigma M = -M - Px = 0 \quad \therefore \quad M = -Px$$

$$U = \frac{1}{2}\int_0^\ell \left(\frac{M^2}{EI} + \cancelto{0}{\frac{F^2}{EA}} + \cancelto{0}{\frac{\kappa V^2}{GA}} \right) dx = \frac{P^2\ell^3}{6EI}$$

$$\frac{\partial U}{\partial P} = \boxed{r_{tip} = \frac{P\ell^3}{3EI}}$$

Resolve the above with the effects of transverse shear.

$$\Sigma F_y = -V - P = 0 \quad \therefore \quad V = -P$$

$$\Sigma M = -M - Px = 0 \quad \therefore \quad M = -Px$$

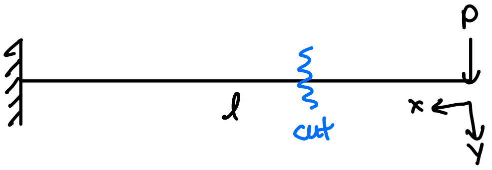
$$U = \frac{1}{2}\int_0^\ell \left(\frac{M^2}{EI} + \cancelto{0}{\frac{F^2}{EA}} + \frac{\kappa V^2}{GA} \right) dx = \frac{P^2\ell^3}{6EI} + \frac{\kappa P^2\ell}{2GA}$$

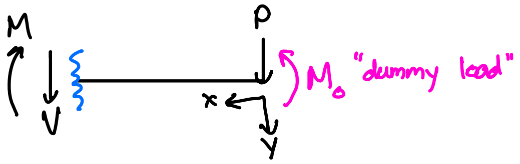
Recall: $G = \frac{E}{2(1+\nu)}$, $A = bh$, $I = \frac{bh^3}{12}$

$$\frac{\partial U}{\partial P} = \boxed{r_{tip} = P\ell\left(\frac{\ell^2}{3EI} + \frac{\kappa}{GA}\right) = \frac{2P\ell}{Ebh}\left[\frac{2\ell^2}{h^2} + (1+\nu)\kappa\right]}$$

the importance of shear depends on the slenderness ratio $s = \ell/h$. The greater s (especially ≥ 10), the less transverse shear matters.

Determine the slope at the end of the beam in the first case (ignore shear).





$$\Sigma M = -M - Px + M_0 = 0 \quad \therefore \quad M = M_0 - Px$$

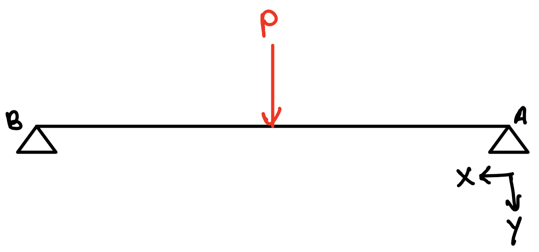
$$U = \frac{1}{2}\int_0^\ell \left(\frac{M^2}{EI} + \cancelto{0}{\frac{F^2}{EA}} + \cancelto{0}{\frac{\kappa V^2}{GA}} \right) dx = \frac{\ell}{6EI}\left(3M_0^2 - 3M_0P\ell + P^2\ell^2\right)$$

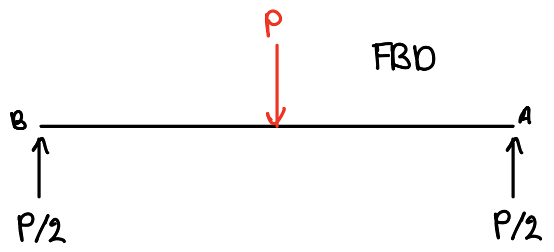
$$\frac{\partial U}{\partial M_0} = \theta_{tip} = \frac{(6\cancelto{0}{M_0} - 3P\ell)}{6EI}$$

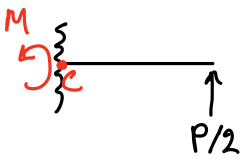
since M_0 is ficticious, our final step is to eliminate it by setting it to zero.

$$\boxed{\theta_{tip} = -\frac{P\ell}{2EI}}$$

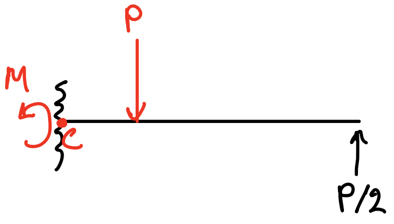
Consider the simply-supported beam loaded as shown below. Use Castigliano's theorem to determine the displacement, u_c, at the beam center.







$$\Sigma M_c = M + \frac{Px}{2} = 0 \qquad \therefore \qquad M = -\frac{Px}{2}$$

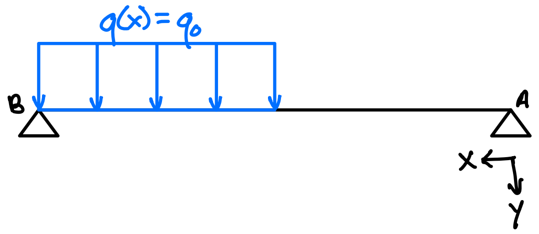


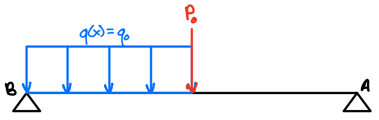
$$\Sigma M_c = M + \frac{Px}{2} - \frac{P}{2}(2x-\ell) = 0 \qquad \therefore \qquad M = \frac{P}{2}(x-\ell)$$

$$U = \frac{1}{2EI}\int_0^{\ell/2}\left(\frac{Px}{2}\right)^2 dx + \frac{1}{2EI}\int_{\ell/2}^{\ell}\left[\frac{P}{2}(x-\ell)\right]^2 dx = \frac{P^2\ell^3}{96EI}$$

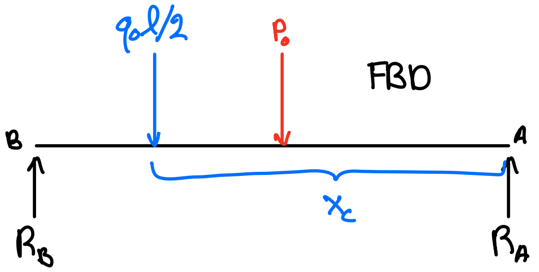
$$u_0 = \frac{dU}{dP} = \frac{P\ell^3}{48EI}$$

Consider the simply-supported beam loaded as shown below. Use Castigliano's theorem to determine the displacement, u_c, at the center of the beam.



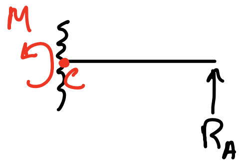


we must add a ficticious load, P_o.

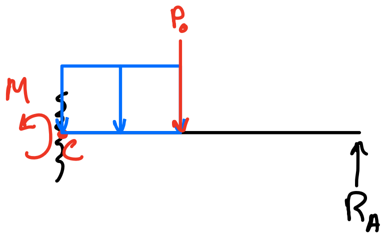


$$x_c=\frac{\int_{\ell/2}^{x} q(s)\,s\,ds}{\int_{\ell/2}^{x} q(s)\,ds}=\frac{\ell+2x}{4}\ ; \quad x=\ell:\quad x_c=\frac{3\ell}{4}$$

$$\left.\begin{aligned}&\Sigma F_y=-R_A-R_B+\frac{q_o\ell}{2}+P_o=0\\ &\Sigma M_A=-R_B\ell+\left(\frac{q_o\ell}{2}\right)\left(\frac{3\ell}{4}\right)+\frac{P_o\ell}{2}=0\end{aligned}\right\}\quad R_A=\frac{1}{8}(4P+q_o\ell)\qquad R_B=\frac{1}{8}(4P+3q_o\ell)$$



$$\Sigma M_c=M+R_Ax=0\qquad \therefore\quad M=-R_Ax$$



$$\Sigma M_c=M+R_Ax-P_o\left(x-\frac{\ell}{2}\right)-q_o\left(x-\frac{\ell}{2}\right)\left(\frac{2x-\ell}{4}\right)=0$$

$$\therefore\quad M=\frac{q_o}{8}(\ell-2x)^2-\frac{P}{2}(\ell-2x)-R_Ax$$

$$U=\frac{1}{2EI}\int_0^{\ell/2}R_A^2x^2\,dx+\frac{1}{2EI}\int_{\ell/2}^{\ell}\left[\frac{q_o}{8}(\ell-2x)^2-\frac{P}{2}(\ell-2x)-R_Ax\right]^2dx=\frac{P^2\ell^3}{192EI}+\frac{P\ell^4q_o}{256EI}+\frac{17q_o^2\ell^5}{15360EI}$$

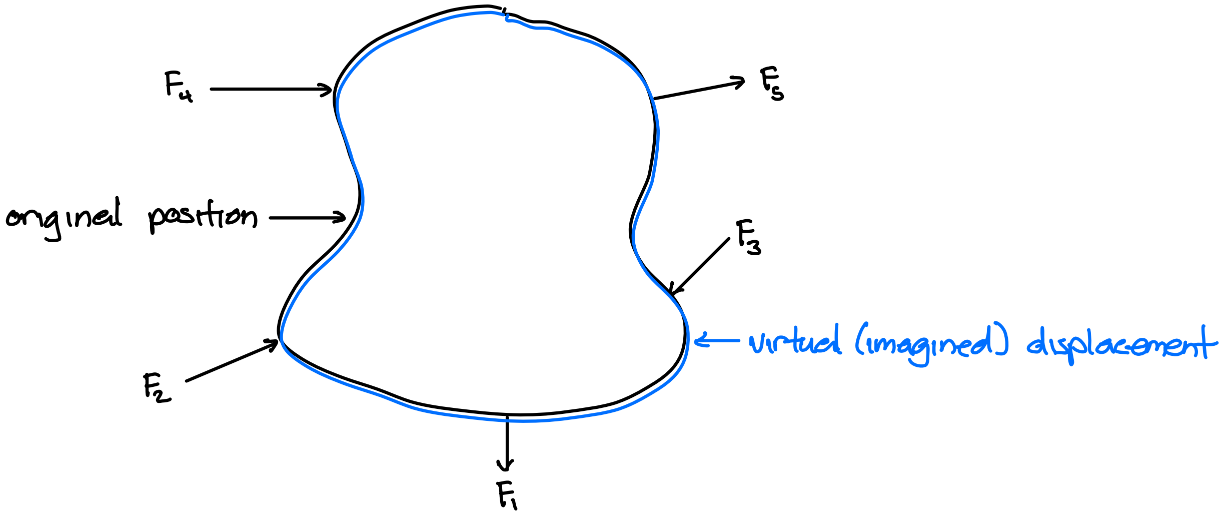
$$u_o=\left.\frac{\partial U}{\partial P}\right|_{P=0}=\frac{q_o\ell^4}{256EI}$$

TOPIC 9:

Principle of Virtual Work and Minimum Potential Energy Theorem

Virtual Work

Consider a rigid body in equilibrium acted on by a set of forces/moments. Now, imagine that the body rigidly displaces without affecting any of the forces that act upon it.



The virtual displacement is "imaginary" meaning that we can choose nearly anything we want; however, these displacements must obey a few rules:

1. infintesimal: allows us to continue using linear elasticity and differentials: du (real differential displacement) and δu (virtual differential displacement).

2. kinematically admissable: virtual displacements must adhere to the same compatibility and constraint conditions as actual displacements

3. instantaneous: reduces complexity by eliminating need to consider virtual velocity/acceleration and the variation of loading in time, i.e., $F(t)$.

$$\delta(uv) = v\delta u + u\delta v \qquad \delta\left(\frac{du}{dx}\right) = \frac{d}{dx}(\delta u)$$

compatibility

$$dF(u,v,t) = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial v}dv + \frac{\partial F}{\partial t}dt \quad \text{but} \quad \delta F(u,v,t) = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial v}dv + \cancelto{0}{\frac{\partial F}{\partial t}dt}$$

due to instantaneity

To develope the principle of virtual work, consider a body acted upon by several forces/moments. Now imagine some virtual displacements. Since, the forces are present, some amount of (virtual) work is done:

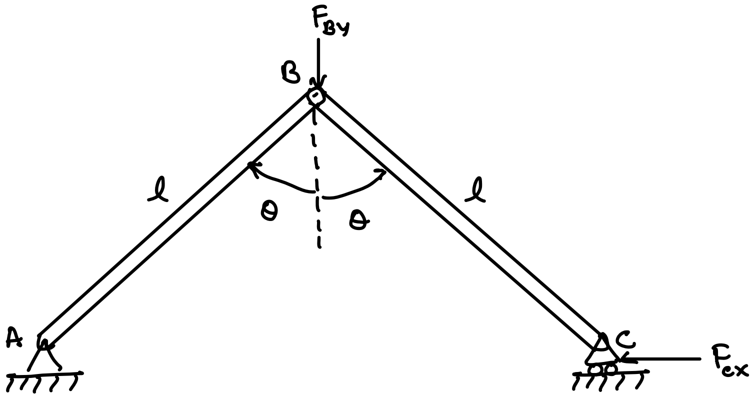
$$\delta W = \bar{F}_1 \cdot \delta\bar{u} + \bar{F}_2 \cdot \delta\bar{u} + \bar{F}_3 \cdot \delta u + \cdots + \bar{F}_n \cdot \delta u = \left(\sum_{i=1}^{n} F_i\right) \cdot \delta\bar{u}$$

If the body is in equillibrium, then $\sum_{i=1}^{n} F_i = 0$ and $\delta W = 0$, which is a statement of the Principle of Virtual Work.

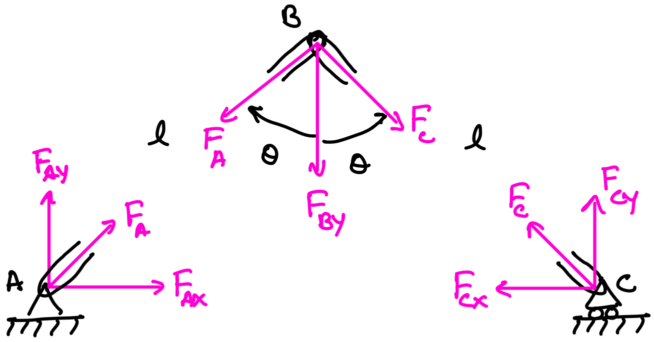
Principle of Virtual Work: for a rigid body to be in static equilibrium, then the virtual work done on the body must be zero.

For rigid bodies, the principle of virtual work offers no advantages over statics. But for a system of interconnected bodies, the principle of virtual work simplifies things by allowing us to skip calculating internal loads and reactions at fixed supports since these do no work.

Problem: Determine the force F_B required to balance the load F_c on an assembly of two pinned-connected rods. Use statics principles.



Method of Joints (from statics class)



Joint C:

$\Sigma F_x = -F_{cx} - F_c \sin\theta = 0 \qquad \therefore \quad F_c = -F_{cx}\csc\theta$

$\Sigma F_y = F_{cy} + F_c\cos\theta = 0 \qquad \therefore \quad F_c = -F_{cy}\sec\theta$

Joint B:

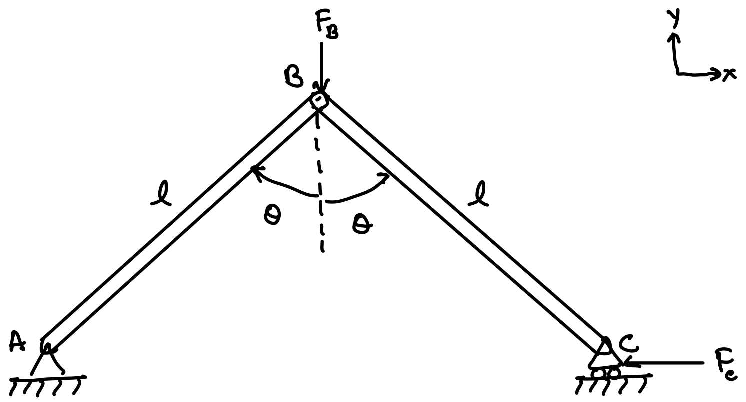
$\Sigma F_x = -F_A\sin\theta + F_c\sin\theta = 0 \qquad \therefore \quad F_c = F_A$

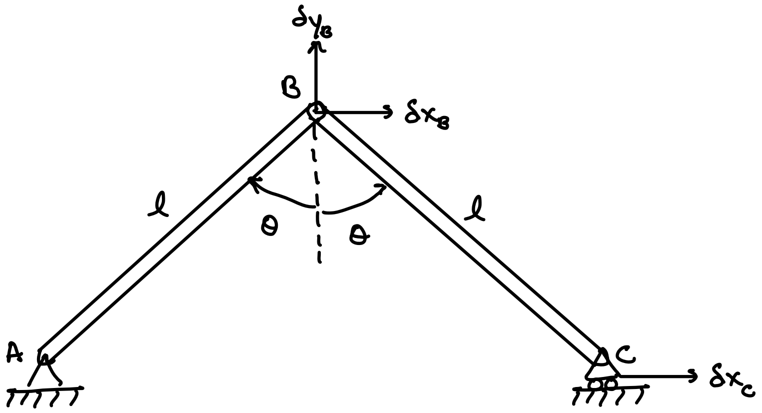
$$\Sigma F_y = -F_A\cos\theta - F_c\cos\theta - F_{By}$$

$$= -2F_c\cos\theta - F_{By}$$

$$= 2F_{cx}\cot\theta - F_{By} = 0 \qquad \therefore \quad \boxed{F_{By} = 2F_{cx}\cot\theta}$$

Problem: Determine the force F_B required to balance the load F_c on an assembly of two pinned-connected rods.





$$\delta W = (0)\delta x_B - F_B \delta y_B - F_c \delta x_c$$

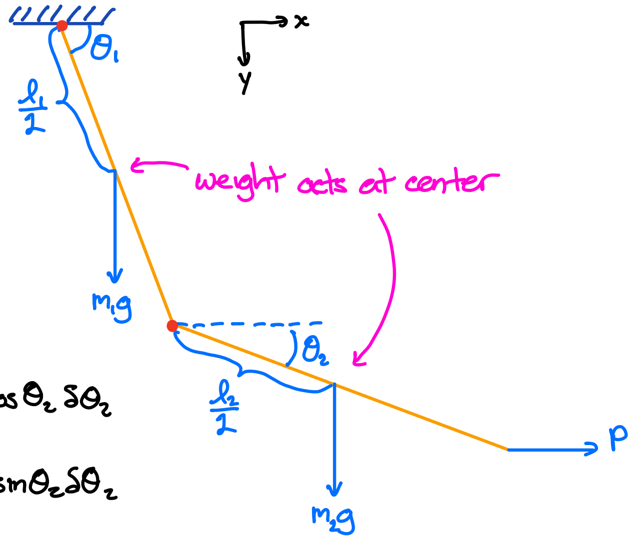
$$y_B = l\cos\theta \qquad \delta y_B = -l\sin\theta\,\delta\theta$$

$$x_c = 2l\sin\theta \qquad \delta x_c = 2l\cos\theta\,\delta\theta$$

$$\delta W = (-F_B\sin\theta + 2F_c\cos\theta)\,l\,\delta\theta = 0 \qquad \therefore\ 2F_c\cos\theta - F_B\sin\theta = 0 \longrightarrow F_c = \frac{1}{2}F_B\tan\theta$$

In statics, we would have to draw free-body diagrams for each member and involve reaction/internal forces. The principle of virtual work greatly simplified things.

Problem: Determine the equilibrium configuration of the double pendulum with horizontal load P applied to the free end.



$$\delta W = m_1 g\,\delta y_1 + m_2 g\,\delta y_2 + P\,\delta x_E = 0$$

$$y_1 = \frac{l_1}{2}\sin\theta_1 \qquad \delta y_1 = \frac{l_1}{2}\cos\theta_1\,\delta\theta_1$$

$$y_2 = l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2 \qquad \delta y_2 = l_1\cos\theta_1\,\delta\theta_1 + \frac{l_2}{2}\cos\theta_2\,\delta\theta_2$$

$$x_E = l_1\cos\theta_1 + l_2\cos\theta_2 \qquad \delta x_E = -l_1\sin\theta_1\,\delta\theta_1 - l_2\sin\theta_2\,\delta\theta_2$$

$$\delta W = \frac{l_1}{2}\underbrace{[(m_1+2m_2)g\cos\theta_1 - 2P\sin\theta_1]}_{\text{must vanish}}\delta\theta_1 + \frac{l_2}{2}\underbrace{(m_2 g\cos\theta_2 - 2P\sin\theta_2)}_{\text{must vanish}}\delta\theta_2 = 0$$

$$(m_1+2m_2)g\cos\theta_1 - 2P\sin\theta_1 = 0 \qquad \therefore\ \theta_1 = \tan^{-1}\left[\frac{(m_1+2m_2)g}{2P}\right]$$

$$m_2 g\cos\theta_2 - 2P\sin\theta_2 \qquad \therefore\ \theta_2 = \tan^{-1}\frac{m_2 g}{2P}$$

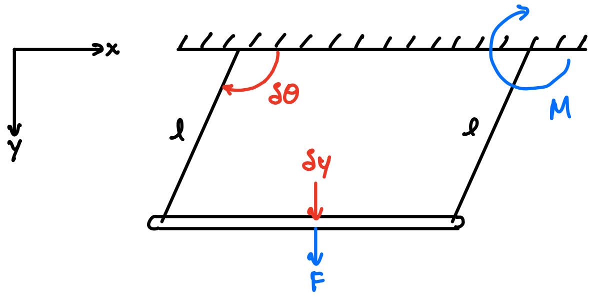
Newton's Method

$$\Sigma M_A = -m_1 g \frac{l_1}{2}\cos\theta_1 - m_2 g\left(l_1\cos\theta_1 + \frac{l_2}{2}\cos\theta_2\right) + P\left(l_1\sin\theta_1 + l_2\sin\theta_2\right) = 0$$

$$= \left(2Pl_1\sin\theta_1 - m_1 g l_1\cos\theta_1 - 2m_2 g l_1\cos\theta_1\right) + \left(2Pl_2\sin\theta_2 - m_2 g l_2\cos\theta_2\right) = 0$$

$$2P\tan\theta_1 = (m_1 + 2m_2)g \qquad\qquad 2P\tan\theta_2 = m_2 g$$

Problem: Determine the value of M which balances the load F such that the system below is in equilibrium.

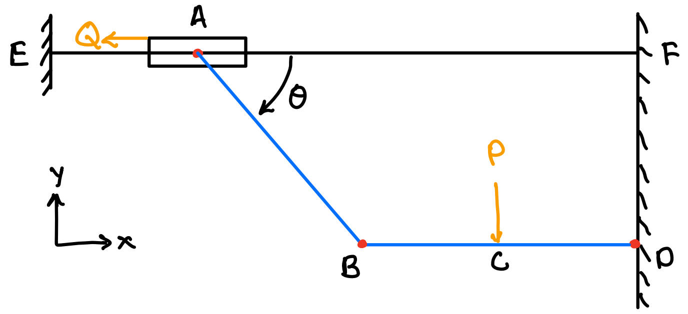


$$\delta W = F\delta y + M\delta\theta$$

$$y = l\sin\theta \qquad \delta y = l\cos\theta\,\delta\theta$$

$$\underbrace{(Fl\cos\theta + M)}_{=0}\delta\theta = 0 \qquad \therefore \qquad \boxed{M = -Fl\cos\theta} \qquad \theta = \cos^{-1}\left(-\frac{M}{Fl}\right)$$

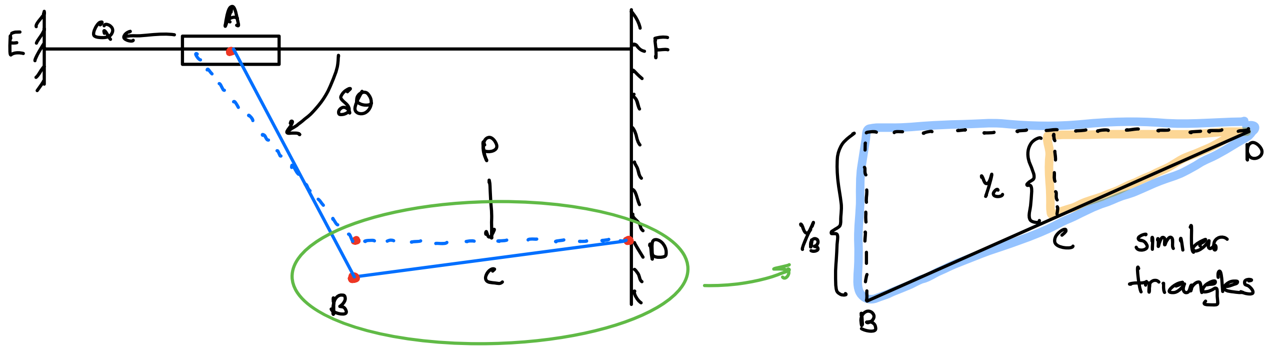
Problem: The collar A slides frictionlessly along the horizontal, rigid rod EF. Collar A is connected to the two-member truss as shown with members AB and BD. Determine the value Q which balances the load P and maintains the system in equilibrium in the configuration shown.



This problem is very similar to the two-member truss problem we did in class. Virtual work for rigid bodies requires we only consider applied loads, not reactions.

$$\delta W = -Q\,\delta x_A - P\,\delta y_c = 0$$

$$x_A = -\ell(1+\cos\theta) \qquad \delta x_A = \ell \sin\theta\,\delta\theta$$



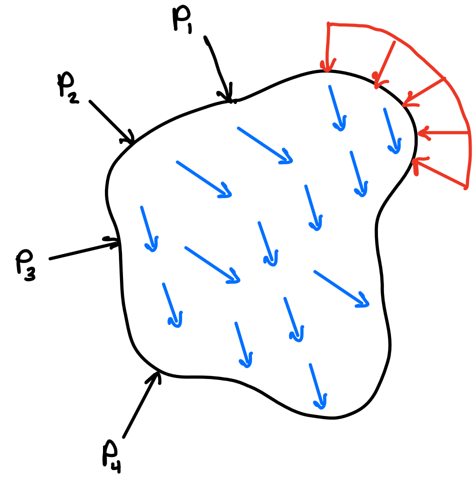
By similar triangles: $\frac{y_B}{\ell} = \frac{y_c}{\ell/2}$ $\quad \therefore\ y_c = \frac{y_B}{2} = -\frac{\ell}{2}\sin\theta \qquad \delta y_c = -\frac{\ell}{2}\cos\theta\,\delta\theta$

$$\delta W = \underbrace{\left(\frac{P}{2}\cos\theta - Q\sin\theta\right)}_{=0 \text{ since } \delta\theta \neq 0}\ell\,\delta\theta = 0$$

$$\therefore\ \boxed{Q = \frac{P}{2}\cot\theta}$$

We have considered virtual work from the perspective of rigid bodies, which must vanish in the case that the system is in equilibrium as the applied forces sum to zero. For deformable bodies, however, some energy is stored in deforming the material, which prevents the virtual work from vanishing.

$$U = \frac{1}{2}\int_V (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\, dV = W$$ strain energy of a deformable body



point forces, $\bar{P}$

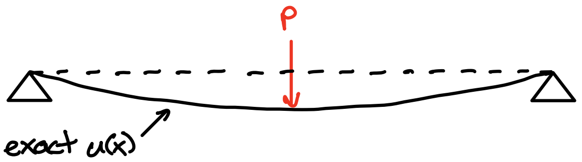
area (surface) forces, $\bar{T}$

body (volumetric) forces, $\bar{b}$

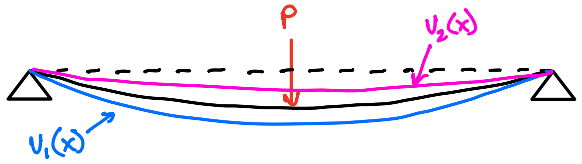
$$U = W = \left(\sum_{i=1}^{n} \bar{P}_i\right)\cdot\bar{u} + \int_A (\bar{T}\, dA)\cdot\bar{u} + \int_V (\bar{b}\, dV)\cdot\bar{u}$$

$$\delta U = \delta W = \left(\sum_{i=1}^{n} \bar{P}_i\right)\cdot\delta\bar{u} + \int_A (\bar{T}\, dA)\cdot\delta\bar{u} + \int_V (\bar{b}\, dV)\cdot\delta\bar{u}$$

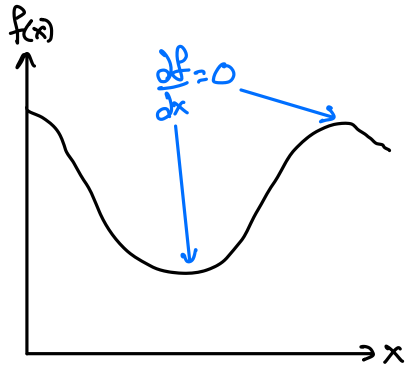
$$\delta\left[\underbrace{U - \left(\sum_{i=1}^{n} \bar{P}_i\right)\cdot u - \int_A (\bar{T}\, dA)\cdot u - \int_V (\bar{b}\, dV)\cdot u}_{\Pi \text{ (total potential energy of a deformable body)}}\right] = 0$$



When a load is applied to an elastic body, the work done by the load balances the potential (strain) energy stored in the body: $U - W = 0$



If we do not know u(x), then we can guess a displacement, v(x), which is a function we make up. If we calculate U and W based on v(x), then we'll find that $U - W \neq 0$. However, we can try to find a form of v(x) such that $\Pi = U - W$ is as close to zero as possible... we want to find the form of v(x) that minimizes Π.



For a function, we find the function minima/maxima by locating the point where the gradient vanishes. A similar concept works for finding the minima/maxima of a functional (a function of a function like $\Pi[v]$ which is a function of the displacement function, v(x)).

Rayleigh - Ritz Rules:

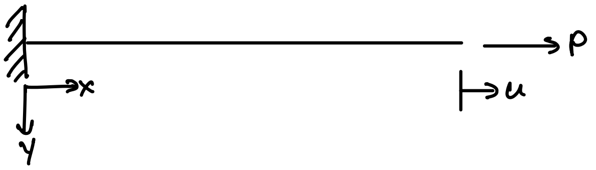
1. the displacement function may take any continuous form: polynomial, trigonometric, logarithmic, a combination, etc.

2. the displacement function must adhere to kinematic constraints (i.e., known displacements from problem definition must not be violated).

3. more terms in displacement function come with more tuning coefficients; more tuning coefficients allows for greater fine tuning of the final solution which improves accuracy.

4. do not eliminate tuning coefficients unless kinematic constraints require it; otherwise you may worsen the accuracy of your final solution.

Problem: Determine the displacement profile, u(x), for a long, narrow cantilever beam with a horizontal tip load. Use the Rayleigh-Ritz Method.



$u = a_0 + a_1 x + a_2 x^2$

$u(0) = a_0 = 0$ $\quad \therefore \quad$ $u(x) = a_1 x + a_2 x^2$

$$U = \frac{1}{2}\int_0^{\ell}\left(\cancel{\frac{M^2}{EI}} + \frac{F^2}{AE} + \cancel{\frac{kV^2}{GA}} + \cancel{\frac{T^2}{GJ}}\right)dx = \frac{1}{2}\int_0^{\ell}\frac{F^2}{AE}dx$$

no bending; no shear; no torsion

$$= \frac{AE}{2}\int_0^{\ell}\left(\frac{du}{dx}\right)^2 dx = \frac{AE}{2}\int_0^{\ell}(a_1 + 2a_2 x)^2 dx = \frac{AE\ell}{2}\left[a_1^2 + 2a_1 a_2 \ell + \frac{4\ell^2}{3}a_2^2\right]$$

$$\sigma = \frac{F}{A} = E\varepsilon = E\frac{du}{dx}$$

$$F = AE\frac{du}{dx}$$

$W = Pu(\ell) = P\ell(a_1 + a_2\ell)$

$\Pi(a_1, a_2) = U - W$

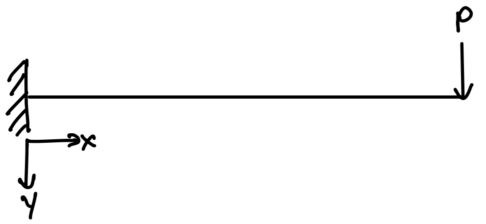
$$\delta\Pi = \frac{\partial\Pi}{\partial a_1}\delta a_1 + \frac{\partial\Pi}{\partial a_2}\delta a_2 = 0$$

$$\left.\begin{aligned} \frac{\partial\Pi}{\partial a_1} &= EA\ell a_1 + EA\ell^2 a_2 - P\ell = 0 \\ \frac{\partial\Pi}{\partial a_2} &= EA\ell^2 a_1 + \frac{4}{3}EA\ell^3 a_2 - P\ell^2 = 0 \end{aligned}\right\} a_1 = \frac{P}{EA};\quad a_2 = 0 \leftarrow$$

extra parameters will go to zero

$u(x) = \frac{P}{EA}x$ exact solution

Problem: Determine the displacement profile, u(x), for a long, narrow cantilever beam with a vertical tip load. Use the Rayleigh-Ritz Method.



$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\left.\begin{aligned} u(0) &= a_0 = 0 \\ \left.\frac{du}{dx}\right|_{x=0} &= a_1 = 0 \end{aligned}\right\} u(x) = a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\frac{d^2u}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$U = \frac{1}{2}\int_0^\ell \left(\frac{M^2}{EI} + \cancel{\frac{F^2}{AE}} + \cancel{\frac{kV^2}{GA}} + \cancel{\frac{T^2}{GJ}} \right) dx = \frac{1}{2}\int_0^\ell \frac{M^2}{EI} dx$$

no axial loads; no shear; no torsion

$$M = EI \frac{d^2u}{dx^2}$$

$$U = \frac{EI}{2}\int_0^\ell (2a_2 + 6a_3 x + 12a_4 x^2)^2 dx$$

$$= \frac{2EI\ell}{5}\left[5a_2^2 + 5a_2\ell(3a_3 + 4\ell a_4) + 3\ell^2(5a_3^2 + 15\ell a_3 a_4 + 12\ell^2 a_4^2)\right]$$

$$W = Pu\Big|_{x=\ell} = P\ell^2(a_2 + a_3\ell + a_4\ell^2)$$

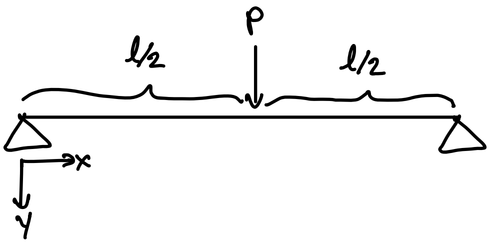
$$\Pi(a_2, a_3, a_4) = U - W \qquad \delta\Pi = \underbrace{\frac{\partial\Pi}{\partial a_2}}_{=0}\delta a_2 + \underbrace{\frac{\partial\Pi}{\partial a_3}}_{=0}\delta a_3 + \underbrace{\frac{\partial\Pi}{\partial a_4}}_{=0}\delta a_4 = 0$$

$$\left.\begin{aligned} \frac{\partial\Pi}{\partial a_2} &= 4\ell a_2 + 6\ell^2 a_3 + 8\ell^3 a_4 - P\ell^2 = 0 \\ \frac{\partial\Pi}{\partial a_3} &= 6\ell^2 a_2 + 12\ell^3 a_3 + 18\ell^4 a_4 - P\ell^3 = 0 \\ \frac{\partial\Pi}{\partial a_4} &= 8\ell^3 a_2 + 18\ell^4 a_3 + \frac{144\ell^5 a_4}{5} - P\ell^4 = 0 \end{aligned}\right\} a_2 = \frac{P\ell}{4} \; ; \; a_3 = -\frac{P}{12} \; ; \; a_4 = 0 \leftarrow$$

extra parameters will go to zero

$$\therefore \; u(x) = \frac{Px^2}{12}(3\ell - x)$$

Problem: Determine the displacement profile, $u(x)$, for a long, narrow simply-supported beam with a vertical load at the mid-span. Use the Rayleigh-Ritz Method. Assume a trigonometric displacement function of order 2.



$$u(x) = a_0 + \sum_{i=1}^{n}\left[a_i \sin\left(\frac{i\pi x}{\ell}\right) + b_i \cos\left(\frac{i\pi x}{\ell}\right)\right]$$ trigonometric function of order n

$$u(x) = a_0 + a_1\sin\left(\frac{\pi x}{\ell}\right) + a_2\sin\left(\frac{2\pi x}{\ell}\right) + b_1\cos\left(\frac{\pi x}{\ell}\right) + b_2\cos\left(\frac{2\pi x}{\ell}\right)$$ trigonometric function of order 2

$u(0) = a_0 = 0$ and $b_i = 0$ every cosine term violates the $u(0)=0$ BC so we immediatly dismiss it. (necessary kinematic constraint)

$$\left.\frac{du}{dx}\right|_{x=\ell/2} = \frac{\pi a_1}{\ell}\cancelto{0}{\cos}\left(\frac{\pi}{2}\right) + \frac{2\pi a_2}{\ell}\cos(\pi) \qquad \therefore\ a_2 = 0$$

we expect the solution to be symmetric about $x = \ell/2$, so $\left.\frac{du}{dx}\right|_{x=\ell/2} = 0$. (non-essential, but helpful kinematic constraint)

$$u(x) = a_1\sin\left(\frac{\pi x}{\ell}\right) \qquad \frac{d^2u}{dx^2} = -a_1\left(\frac{\pi}{\ell}\right)^2\sin\left(\frac{\pi x}{\ell}\right)$$

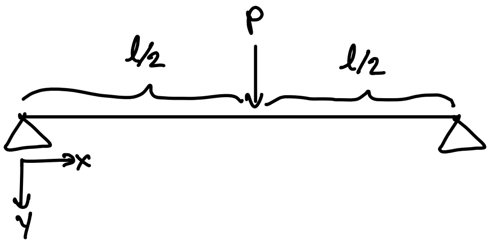
$$U = \frac{EI}{2}\left(\frac{\pi}{\ell}\right)^4 a_1^2\int_0^{\ell}\sin^2\left(\frac{\pi x}{\ell}\right)dx = \frac{EI\pi^4 a_1^2}{4\ell^3}$$

$$W = Pu(\ell/2) = Pa_1$$

$$\frac{\partial\Pi}{\partial a_1} = \frac{EI\pi^4 a_1}{2\ell^3} - P = 0 \qquad \therefore \qquad a_1 = \frac{2P\ell^3}{EI\pi^4}$$

$$u(x) = \frac{2P\ell^3}{EI\pi^4}\sin\left(\frac{n\pi x}{\ell}\right)$$ not the exact solution, but within a few percent

Problem: Determine the displacement profile, $u(x)$, for a long, narrow simply-supported beam with a vertical load at the mid-span. Use the Rayleigh-Ritz Method. Assume a polynomial displacement function of order 4.



$$u(x) = \sum_{i=0}^{3} a_i x^i (\ell - x) = a_0(\ell - x) + a_1 x(\ell - x) + a_2 x^2(\ell - x) + a_3 x^3(\ell - x)$$

$$= a_0\ell - (a_0 - a_1\ell)x - (a_1 - a_2\ell)x^2 - (a_2 - a_3\ell)x^3 - a_3x^4 \qquad u(\ell) = 0$$ by construction

$$u(0) = a_0 = 0 \qquad \therefore \qquad a_0 = 0$$

$$\left.\frac{du}{dx}\right|_{x=\ell/2} = \frac{\ell^2}{4}(a_2 + a_3\ell) = 0 \qquad \therefore \qquad a_3 = -\frac{a_2}{\ell}$$

If we make this substitution in the displacement function, we can simplify the remaining calculations.

$$u(x) = a_1\ell x - (a_1 - a_2\ell)x^2 - \frac{a_2}{\ell}x^3(2\ell - x)$$

$$\frac{d^2u}{dx^2} = -2a_1 + \frac{2a_2}{\ell}(\ell^2 - 6\ell x + 6x^2)$$

$$U = \frac{2EI}{\ell^2}\int_0^\ell [a_1\ell - a_2(\ell^2 - 6\ell x + 6x^2)]^2 dx = \frac{2EI\ell}{5}(5a_1^2 + a_2^2\ell^2)$$

$$W = Pu(\ell/2) = \frac{P\ell^2}{16}(4a_1 + a_2\ell)$$

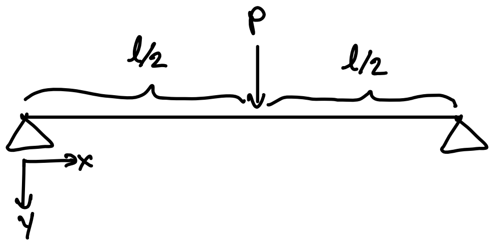
$$\Pi = U - W$$

$$\delta\Pi = 0 \quad \text{requires} \quad \frac{\partial\Pi}{\partial a_1} = \frac{\partial\Pi}{\partial a_2} = 0$$

$$\left.\begin{aligned} \frac{\partial\Pi}{\partial a_1} &= 4EI\ell a_1 - \frac{P\ell^2}{4} = 0 \\ \frac{\partial\Pi}{\partial a_2} &= \frac{4EI\ell^3}{5}a_2 - \frac{P\ell^3}{16} = 0 \end{aligned}\right\} \quad a_1 = \frac{P\ell}{16EI}; \quad a_2 = \frac{5P}{64EI}; \quad a_3 = -\frac{a_2}{\ell} = -\frac{5P}{64EI\ell}$$

$$u(x) = \frac{Px(\ell - x)(4\ell^2 + 5\ell x - 5x^2)}{64EI\ell}$$ not the exact solution, but within a few percent

Problem: Determine the displacement profile, u(x), for a long, narrow simply-supported beam with a vertical load at the mid-span. Use the Rayleigh-Ritz Method. Assume a polynomial displacement function of order 4.



$$u(x) = \sum_{i=0}^{3} a_i x^i (\ell - x) = a_0(\ell - x) + a_1 x(\ell - x) + a_2 x^2 (\ell - x) + a_3 x^3 (\ell - x)$$

$$= a_0 \ell - (a_0 - a_1 \ell)x - (a_1 - a_2 \ell)x^2 - (a_2 - a_3 \ell)x^3 - a_3 x^4 \qquad u(\ell) = 0$$ by construction

$$u(0) = a_0 = 0 \qquad \therefore \qquad a_0 = 0$$

$$\left.\frac{du}{dx}\right|_{x=\ell/2} = \frac{\ell^2}{4}(a_2 + a_3 \ell) = 0 \qquad \therefore \qquad a_3 = -\frac{a_2}{\ell}$$

If we make this substitution in the displacement function, we can simplify the remaining calculations. I forgot to do this which results in more work for the same answer!

$$\frac{d^2u}{dx^2} = -2a_1 + 2a_2(\ell - 3x) + 6a_3 x(\ell - 2x)$$

$$U = \frac{EI}{2}\int_0^{\ell} [2a_1 - 2a_2(\ell - 3x) - 6a_3 x(\ell - 2x)]^2 dx$$

$$= \frac{EI}{2}\left(4\ell a_1^2 + 4\ell^3 a_2^2 + \frac{24\ell^5}{5} a_3^2 + 4\ell^2 a_1 a_2 + 4\ell^3 a_1 a_3 + 8\ell^4 a_2 a_3\right)$$

$$W = Pu(\ell/2) = \frac{P}{16}(4\ell^2 a_1 + 2\ell^3 a_2 + \ell^4 a_3)$$

$$\Pi = U - W$$

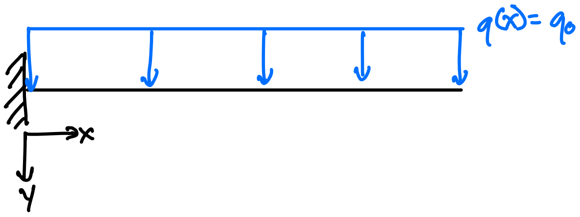
$$\delta\Pi = 0 \qquad \text{requires} \qquad \frac{\partial \Pi}{\partial a_1} = \frac{\partial \Pi}{\partial a_2} = \frac{\partial \Pi}{\partial a_3} = 0$$

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial a_1} &= 2EI\ell[2a_1 + (a_2 + a_3\ell)\ell] - \frac{P\ell^2}{4} = 0 \\ \frac{\partial \Pi}{\partial a_2} &= 2EI\ell^2[a_1 + 2(a_2 + a_3\ell)\ell] - \frac{P\ell^3}{8} = 0 \\ \frac{\partial \Pi}{\partial a_3} &= \frac{EI\ell^3}{5}[10a_1 + 4(5a_2 + 6a_3\ell)\ell] - \frac{P\ell^4}{16} = 0 \end{aligned}\right\} a_1 = \frac{P\ell}{16EI}; \quad a_2 = \frac{5P}{64EI}; \quad a_3 = -\frac{5P}{64EI\ell}$$

$$u(x) = \frac{Px(\ell - x)(4\ell^2 + 5\ell x - 5x^2)}{64EI\ell}$$

not the exact solution, but within a few percent

Problem: Determine the displacement profile, $u(x)$, for a long, narrow cantilever with a uniform distributed load. Use the Rayleigh-Ritz Method.



$$\left.\begin{aligned}
&u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\
&\left.\begin{aligned} &u(0) = a_0 = 0 \\ &\left.\frac{du}{dx}\right|_{x=0} = a_1 = 0 \end{aligned}\right\} u(x) = a_2 x^2 + a_3 x^3 + a_4 x^4 \\
&\frac{d^2 u}{dx^2} = 2a_2 + 6a_3 x + 12 a_4 x^2 \\
&U = \frac{EI}{2}\int_0^{\ell} (2a_2 + 6a_3 x + 12 a_4 x^2)^2 dx \\
&\quad = \frac{2EI\ell}{5}[5a_2^2 + 5a_2\ell(3a_3 + 4\ell a_4) + 3\ell^2(5a_3^2 + 15\ell a_3 a_4 + 12\ell^2 a_4^2)]
\end{aligned}\right\} \text{same as above}$$

$$W = \int_0^{\ell} q(x)u(x)\,dx = q_0\left(\frac{\ell^3}{3}a_2 + \frac{\ell^4}{4}a_3 + \frac{\ell^5}{5}a_4\right)$$

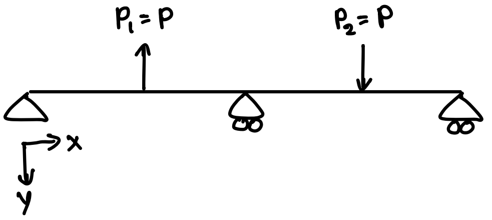
$$\Pi = U - W$$

$$\delta\Pi = \delta(U - W) = \frac{\partial \Pi}{\partial a_2}\delta a_2 + \frac{\partial \Pi}{\partial a_3}\delta a_3 + \frac{\partial \Pi}{\partial a_4}\delta a_4 = 0$$

$$\left.\begin{aligned}
&\frac{\partial \Pi}{\partial a_2} = 4\ell a_2 + 6\ell^2 a_3 + 8\ell^3 a_4 - \frac{q_0 \ell^3}{3} = 0 \\
&\frac{\partial \Pi}{\partial a_3} = 6\ell^2 a_2 + 12\ell^3 a_3 + 18\ell^4 a_4 - \frac{q_0 \ell^4}{4} = 0 \\
&\frac{\partial \Pi}{\partial a_4} = 8\ell^3 a_2 + 18\ell^4 a_3 + \frac{144\ell^5 a_4}{5} - \frac{q_0 \ell^5}{5} = 0
\end{aligned}\right\} a_2 = \frac{\ell^2 q}{4EI};\; a_3 = -\frac{\ell q}{6EI};\; a_4 = \frac{q}{24EI}$$

$$\therefore\; u(x) = \frac{qx^2}{24EI}(6\ell^2 - 4\ell x + x^2)$$ exact solution

Use a fifth-degree polynomial and the Rayleigh-Ritz method to determine the displacement profile of the simply-supported beam loaded as shown below.



$$u(x) = a_0 + a_1 x^2 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$u(0) = a_0 = 0 \qquad \therefore \qquad a_0 = 0$$

$$u(\ell/2) = \frac{\ell}{32}(16a_1 + 8a_2\ell + 4a_3\ell^2 + 2a_4\ell^3 + a_5\ell^4) = 0 \qquad \therefore \qquad a_5 = -\frac{2}{\ell^4}(8a_1 + 4a_2\ell + 2a_3\ell^2 + a_4\ell^3)$$

$$u(x) = a_1 x^2 + a_2 x^2 + a_3 x^3 + a_4 x^4 - \frac{2}{\ell^4}(8a_1 + 4a_2\ell + 2a_3\ell^2 + a_4\ell^3)x^5$$

$$u(\ell) = -\ell(15a_1 + 7a_2\ell + 3a_3\ell^2 + a_4\ell^3) = 0 \qquad \therefore \qquad a_4 = -\frac{1}{\ell^3}(15a_1 + 7a_2\ell + 3a_3\ell^2)$$

$$u(x) = \frac{a_1 x}{\ell^4}(\ell^4 - 15x^3\ell + 14x^4) + \frac{a_2 x^2}{\ell^3}(\ell^3 - 7x^2\ell + 6x^3) + \frac{a_3 x^3}{\ell^2}(\ell - 2x)(\ell - x)$$

At this point we have accounted for all kinematic constraints and may proceed to solve the problem of $\delta\Pi = 0$; however, we may take advantage of (anti-)symmetry to define more BCs. Every BC utilized is a coefficient eliminated, which reduces the number of coefficients to solve for.

$$\left.\frac{du}{dx}\right|_{x=\ell/4} = \frac{43}{128}a_1 + \frac{23\ell}{128}a_2 + \frac{5\ell^2}{128}a_3 = 0 \qquad \therefore \qquad a_3 = -\frac{1}{5\ell^2}(43a_1 + 23a_2\ell)$$

$$u(x) = \frac{a_1}{5\ell^4}x(\ell - 2x)(\ell - x)(5\ell^2 + 15\ell x - 8x^2) + \frac{a_2}{5\ell^3}x^2(5\ell - 8x)(\ell - 2x)(\ell - x)$$

$$\left.\frac{du}{dx}\right|_{x=3\ell/4} = -\frac{1}{20}(7a_1 - 3a_2\ell) = 0 \qquad \therefore \qquad a_2 = \frac{7a_1}{3\ell}$$

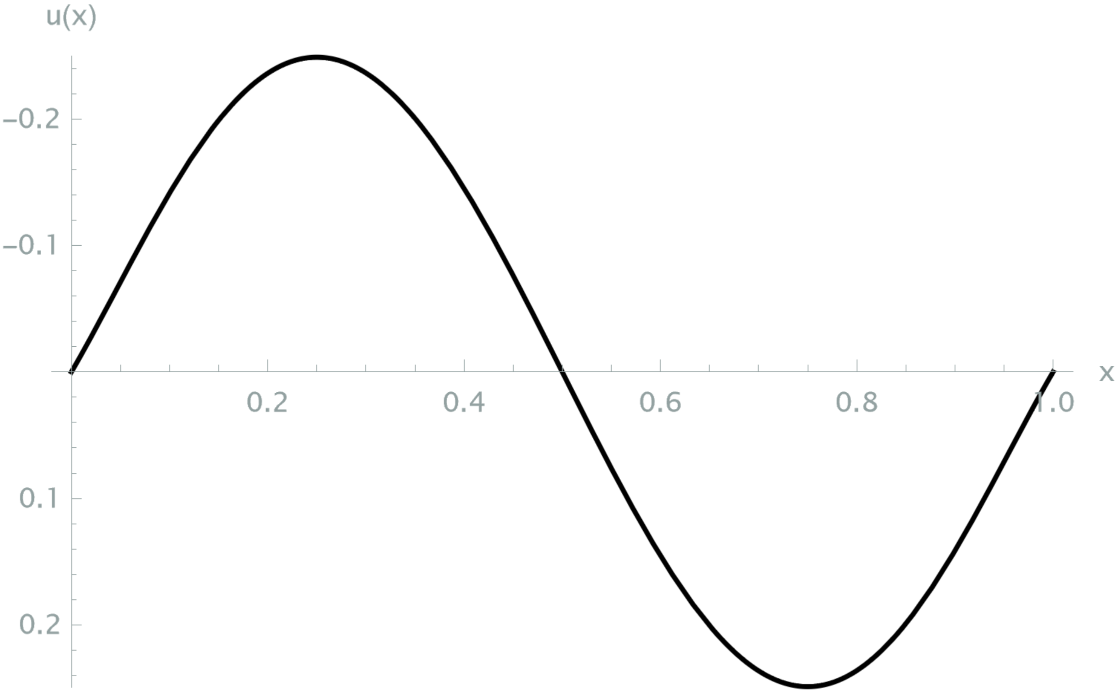
$$u(x) = \frac{a_1 x}{3\ell^4}(3\ell^4 + 7\ell^3 x - 58\ell^2 x^2 + 80\ell x^3 - 32x^4)$$

$$U = \frac{1}{2EI}\int_0^{\ell}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx = \frac{890}{63EI\ell}a_1^2$$

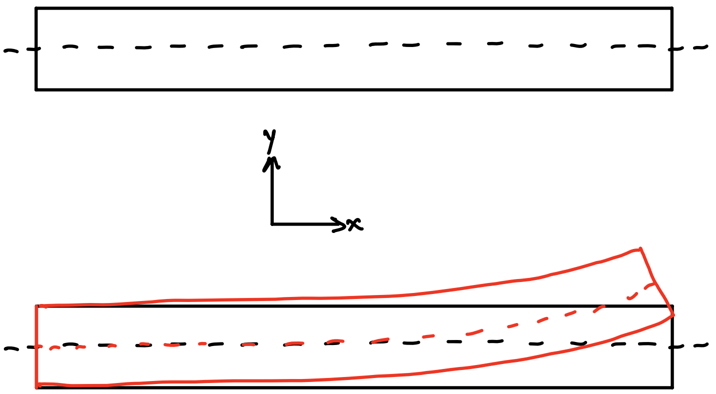
$$W = -P_1 u(\ell/4) + P_2 u(3\ell/4) = -\frac{3P\ell}{8}a_1$$

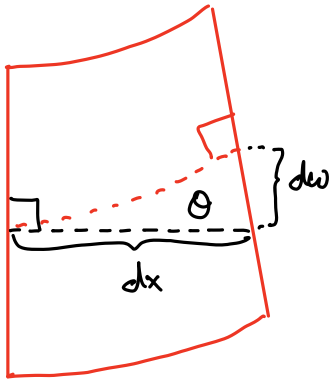
$$\delta\Pi = \delta(U-W) = \frac{\partial \Pi}{\partial a_1}\delta a_1 = \left(\frac{1780}{63EI\ell}a_1 + \frac{3P\ell}{8}\right)\delta a_1 = 0 \qquad \therefore \qquad a_1 = -\frac{189EIP\ell^2}{14240}$$

Let $P=10$, $\ell=1$, and $EI=10$, then:



Euler - Bernoulli Beam (slender beam with transverse loading)





$$\tan\theta \approx \frac{dw}{dx}$$

$$\theta = \frac{dw}{dx}$$

$$\gamma_{xy} = \frac{du}{dy} + \frac{dw}{dx} \approx 0$$

$$du = -\frac{dw}{dx}\,dy = -\theta dy$$

$$u = -\theta y$$

$$\varepsilon_x = \frac{du}{dx} = -y\frac{\partial^2 \omega}{\partial x^2}$$

$$\sigma_x = E\varepsilon_x = -Ey\frac{\partial^2 \omega}{\partial x^2}$$

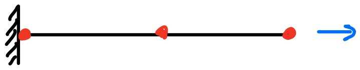
$I_z = \int y^2 dA$ area moment of inertia w.r.t. z-axis

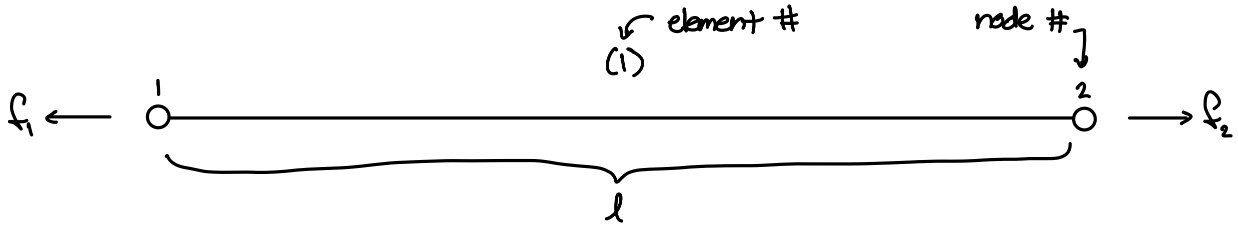
$$dF = \sigma_x dA = -Ey\frac{\partial^2 \omega}{\partial x^2}dA$$

$$M = \int dF y = -E\int y^2 \frac{\partial^2 \omega}{\partial x^2} dA = -EI_z\frac{\partial^2 \omega}{\partial x^2} = \underline{-EI_z\frac{\partial \theta}{\partial x}}$$

TOPIC 10:
Introduction to the Finite Element Method

rods: supports axial loads only.





$$U = \frac{1}{2}\int_0^{\ell}\frac{(f_2-f_1)^2}{AE}dx = \frac{\ell}{2AE}(f_2-f_1)^2 = \frac{AE}{2\ell}(u_2-u_1)^2$$

$$\sigma_x = \frac{F}{A} = E\frac{du}{dx} = \frac{Eu}{\ell} \quad \therefore \quad F = \frac{AEu}{\ell}$$

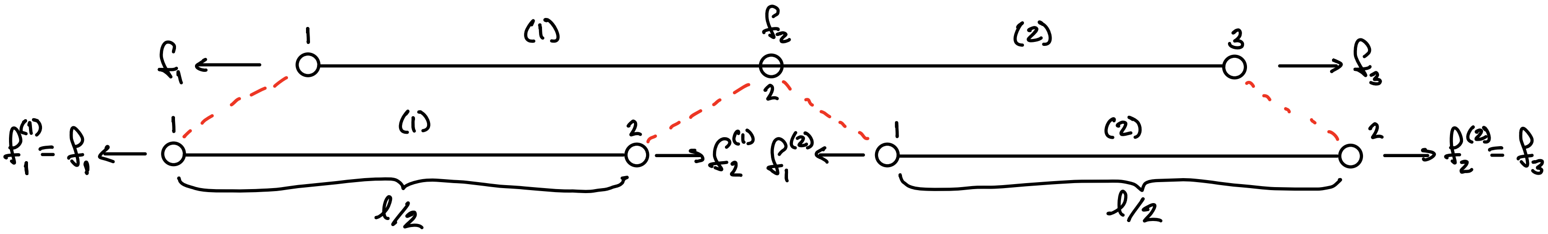
$$\sum F = f_2 - f_1 = \frac{AE}{\ell}(u_2-u_1)$$

Castigliano's second theorem:

$$\left.\begin{aligned} u_1 &= \frac{\partial U}{\partial f_1} = \frac{(f_1-f_2)\ell}{AE} \\ u_2 &= \frac{\partial U}{\partial f_2} = \frac{(f_2-f_1)\ell}{AE} \end{aligned}\right\} \quad \underset{\bar{u}}{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} = \underbrace{\frac{\ell}{AE}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{K^{-1}} \underset{\bar{f}}{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}$$

Castigliano's first theorem:

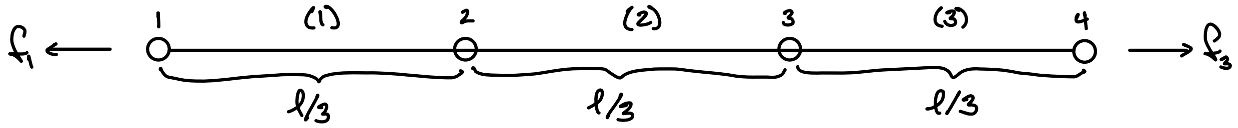
$$\left.\begin{aligned} \frac{\partial U}{\partial u_1} &= \frac{AE}{\ell}(u_1-u_2) = f_1 \\ \frac{\partial U}{\partial u_2} &= \frac{AE}{\ell}(u_2-u_1) = f_2 \end{aligned}\right\} \quad \underbrace{\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{K} \underset{\bar{u}}{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} = \underset{\bar{f}}{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}$$



$$\sum F^{(1)} = f_2^{(1)} - f_1^{(1)} = \frac{2AE}{\ell}(u_2-u_1) \qquad \sum F^{(2)} = f_2^{(2)} - f_1^{(2)} = \frac{2AE}{\ell}(u_3-u_2)$$

$$U = \frac{2AE}{\ell^2}\int_0^{\ell/2}(u_2-u_1)^2dx + \frac{2AE}{\ell^2}\int_{\ell/2}^{\ell}(u_3-u_2)^2dx = \frac{AE}{\ell}\left[(u_1-u_2)^2 + (u_2-u_3)^2\right]$$

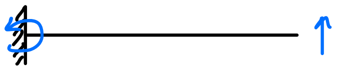
$$\left.\begin{aligned} \frac{\partial U}{\partial u_1} &= \frac{2AE}{\ell}(u_1-u_2) = f_1 \\ \frac{\partial U}{\partial u_2} &= \frac{2AE}{\ell}(-u_1+2u_2-u_3) = f_2 \\ \frac{\partial U}{\partial u_3} &= \frac{2AE}{\ell}(-u_2+u_3) = f_3 \end{aligned}\right\} \quad \underbrace{\frac{2AE}{\ell}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{K} \underset{\bar{u}_L}{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}} = \underset{\bar{f}}{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}$$



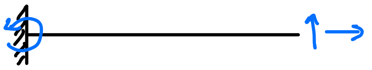
$$U = \frac{9AE}{2l^2}\int_0^{l/3}(u_2-u_1)^2dx + \frac{9AE}{2l^2}\int_{l/3}^{2l/3}(u_3-u_2)^2dx + \frac{9AE}{2l^2}\int_{2l/3}^{l}(u_4-u_3)^2dx = \frac{3AE}{2l}\left[(u_2-u_1)^2+(u_3-u_2)^2+(u_4-u_3)^2\right]$$

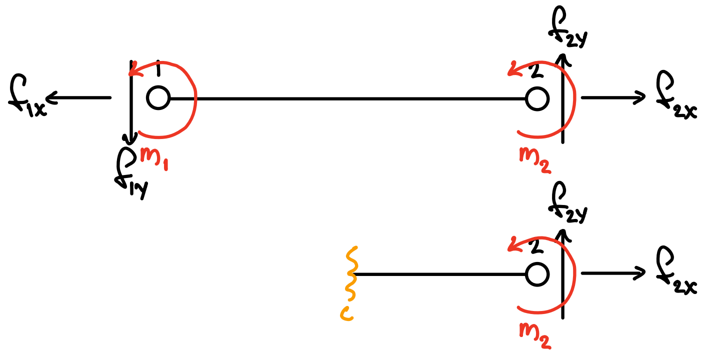
$$\left.\begin{aligned}
\frac{\partial U}{\partial u_1} &= \frac{3AE}{l}(u_1-u_2) = f_1\\
\frac{\partial U}{\partial u_2} &= \frac{3AE}{l}(-u_1+2u_2-u_3) = f_2\\
\frac{\partial U}{\partial u_3} &= \frac{3AE}{l}(-u_2+2u_3-u_4) = f_3\\
\frac{\partial U}{\partial u_4} &= \frac{3AE}{l}(-u_3+u_4) = f_4
\end{aligned}\right\}
\quad
\underbrace{\frac{3AE}{l}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}}_{K}
\underbrace{\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}}_{\bar{u}}
=
\underbrace{\begin{bmatrix}f_1\\f_2\\f_3\\f_4\end{bmatrix}}_{\bar{f}}$$

beams: supports bending and transverse loads.



truss: supports bending, transverse, and axial loads.





$$\Sigma M = m_1 + m_2 + f_{1y}l = 0$$

$$m_1 = -(m_2 + f_{1y}l)$$

Let's fix node 1 and apply f_{2x}, f_{2y}, and m_2 at node 2:

$$U = \frac{1}{2}\int_0^l\left(\frac{M^2}{EI} + \frac{F^2}{AE}\right)dx$$ assuming shear is negligible

$$\Sigma M_c = M + m_2 + f_{2y}x = 0 \qquad \therefore\ M = -(m_2 + f_{2y}x)$$

$$\Sigma F_x = -F + f_{2x} = 0 \qquad \therefore\ F = f_{2x}$$

$$= \frac{1}{2}\int_0^l\left[\frac{(m_2+f_{2y}x)^2}{EI} + \frac{f_{2x}^2}{AE}\right]dx$$

$$= \frac{l}{2}\left(\frac{f_{2x}^2}{AE} + \frac{f_{2y}^2 l^2}{3EI} + \frac{m_2^2}{EI} + \frac{f_{2y}m_2 l}{EI}\right)$$

$$\left.\begin{aligned} u_{1x} &= \frac{\partial U}{\partial f_{2x}} = \frac{l}{AE} f_{2x} \\ u_{2y} &= \frac{\partial U}{\partial f_{2y}} = \frac{l^3}{3EI} f_{2y} + \frac{l^2}{2EI} m_2 \\ \theta_2 &= \frac{\partial U}{\partial m_2} = \frac{l^2}{2EI} f_{2y} + \frac{l}{EI} m_2 \end{aligned}\right\} \begin{bmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{2y} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_{2x} \\ f_{2y} \\ m_2 \end{bmatrix}$$

Notice if $u_{2x} \neq 0$, but $u_{2y} = 0$ and $\theta_2 = 0$, then

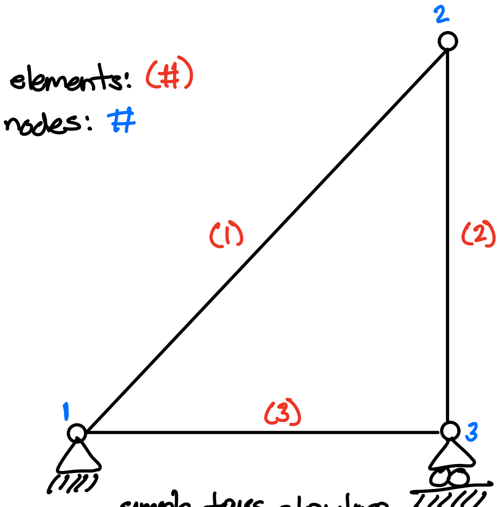
$$f_{2x} = \frac{AE}{l} u_{2x}$$ an axial displacement requires an axial force

if $u_{2y} \neq 0$, but $u_{2y} = 0$ and $\theta_2 = 0$, then

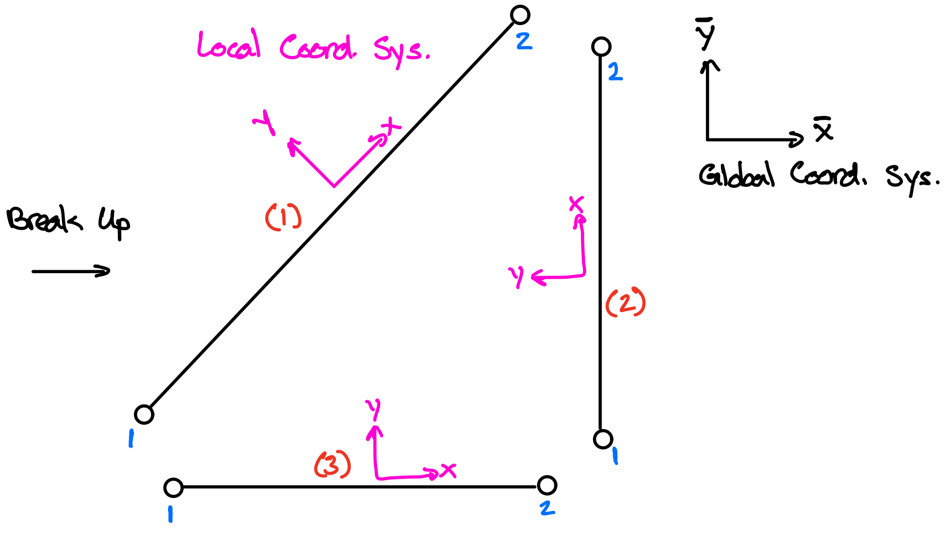
$$\left.\begin{aligned} f_{2y} &= \frac{12EI}{l^3} u_{2y} \\ m_2 &= -\frac{6EI}{l^2} u_{2y} \end{aligned}\right\}$$ a transverse displacement requires both a transverse force and a moment

If we take into account the forces/moments at node 1, then it can be shown that:

$$\underbrace{\begin{bmatrix} \frac{AE}{l} & 0 & 0 & -\frac{AE}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{AE}{l} & 0 & 0 & \frac{AE}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} u_{1x} \\ u_{1y} \\ \theta_1 \\ u_{2x} \\ u_{2y} \\ \theta_2 \end{bmatrix}}_{\bar{u}} = \underbrace{\begin{bmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{bmatrix}}_{\bar{f}}$$



simple truss structure



FE Model

The rod FE matrix equations are based on the 1D governing equation $EA\frac{\partial^2 u}{\partial x^2}+f=0$. In the local coordinate system, nodes displace in either the x- or y-directions, but in the global coordinate system, the nodes may displace in both $\bar{x}$- and $\bar{y}$-directions. All elements in a FE model must adhere to the same (global) coordinate system.

$$K^e = \frac{AE}{\ell^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \xrightarrow[u_{1y}\text{ and }u_{2y}]{\text{expand to include}} \frac{AE}{\ell^e}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\bar{u}^e = \begin{bmatrix}u_{1x}\\ u_{2x}\end{bmatrix} \xrightarrow[u_{1y}\text{ and }u_{2y}]{\text{expand to include}} \begin{bmatrix}u_{1x}\\ u_{1y}\\ u_{2x}\\ u_{2y}\end{bmatrix} \qquad \bar{f}^e = \begin{bmatrix}f_{1x}\\ f_{2x}\end{bmatrix} \xrightarrow[f_{1y}\text{ and }f_{2y}]{\text{expand to include}} \begin{bmatrix}f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\end{bmatrix}$$

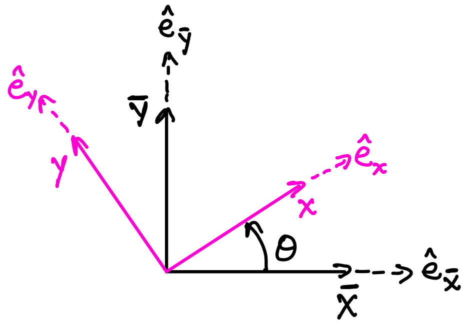
$$\bar{u}_1 = \begin{bmatrix}u_{1x}\\ u_{1y}\end{bmatrix} = \overbrace{u_{1x}\hat{e}_x + u_{1y}\hat{e}_y}^{\text{local}} = \overbrace{u_{1\bar{x}}\hat{e}_{\bar{x}} + u_{1\bar{y}}\hat{e}_{\bar{y}}}^{\text{global}}$$

$$\bar{u}_1\cdot\hat{e}_x = u_{1x}(\cancelto{1}{\hat{e}_x\cdot\hat{e}_x}) + u_{1y}(\cancelto{0}{\hat{e}_y\cdot\hat{e}_x}) = u_{1x} = u_{1\bar{x}}(\hat{e}_{\bar{x}}\cdot\hat{e}_x) + u_{1\bar{y}}(\hat{e}_{\bar{y}}\cdot\hat{e}_x)$$

$$= u_{1\bar{x}}\cos\theta + u_{1\bar{y}}\sin\theta$$

$$\bar{u}_1\cdot\hat{e}_y = u_{1x}(\cancelto{0}{\hat{e}_x\cdot\hat{e}_y}) + u_{1y}(\cancelto{1}{\hat{e}_y\cdot\hat{e}_y}) = u_{1y} = u_{1\bar{x}}(\hat{e}_{\bar{x}}\cdot\hat{e}_y) + u_{1\bar{y}}(\hat{e}_{\bar{y}}\cdot\hat{e}_y)$$

$$= -u_{1\bar{x}}\sin\theta + u_{1\bar{y}}\cos\theta$$



The same transformations can be done for $\bar{u}_2$ as well as be formulated to account for the z-coord. Together, the 2D local-global transformation is:

$$\overset{u^L}{\underbrace{\begin{bmatrix}u_{1x}\\u_{1y}\\u_{2x}\\u_{2y}\end{bmatrix}}_{\text{local displacement vector}}} = \underbrace{\begin{bmatrix}c&s&0&0\\-s&c&0&0\\0&0&c&s\\0&0&-s&c\end{bmatrix}}_{R \text{ rotation matrix}} \overset{u^G}{\underbrace{\begin{bmatrix}u_{1\bar{x}}\\u_{1\bar{y}}\\u_{2\bar{x}}\\u_{2\bar{y}}\end{bmatrix}}_{\text{global displacement vector}}}$$

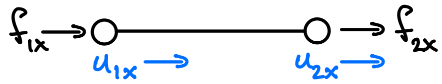
All along we've written K^e and $\bar{f}^e$ in terms of the local $\bar{u}^L$ displacements. We must move to the global system.

$$U^e = \frac{1}{2}(\bar{u}^L)^T K^e \bar{u}^L = \frac{1}{2}(R\bar{u}^G)^T K^e R\bar{u}^G = \frac{1}{2}(\bar{u}^G)^T \underbrace{R^T K^e R}_{K^e \text{ in global system}} \bar{u}^G = \frac{1}{2}(\bar{u}^G)^T K^e \bar{u}^G$$

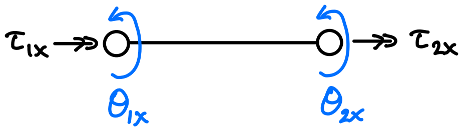
$$K^e = \frac{E^e A^e}{l^e}\begin{bmatrix}c^2 & sc & -c^2 & -sc\\ sc & s^2 & -sc & -s^2\\ -c^2 & -sc & c^2 & sc\\ -sc & -s^2 & sc & s^2\end{bmatrix}$$

where $c = \cos\theta$; $s = \sin\theta$

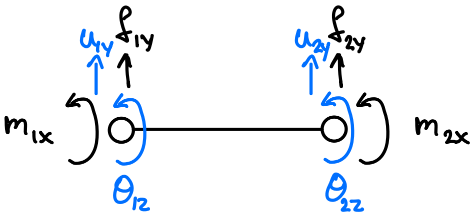
Similarly: $\underset{\text{global}}{\bar{f}^e} = R^T \underset{\text{local}}{\bar{f}^L}$



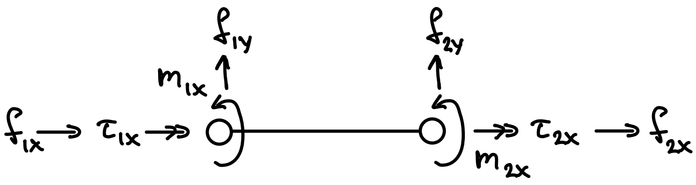
rod: supports axial loads



shaft: supports torsional loads



beam: supports bending moments and shear loads



truss: supports bending moments, and torsional, shear, and axial loads

TOPIC 11:

Indicial Notation and Summation

Indicial Notation and Summation

3.1 The Summation Convention

Einstein summation is a notational convention for simplifying expressions including summations of matrices and tensors, in general. There are essentially three rules of Einstein summation notation, namely:

1. Indices repeated within a term are implicitly summed over.
2. Each term must contain identical non-repeated indices.
3. Each index can appear at most twice in any term.

Rule 1 on the above list can be employed to greatly simplify and shorten equations involving tensors. For example, using the summation convention, the dot product of vectors **u** and **v** is written as

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \equiv \sum_i u_i v_i.$$

Similarly, the double dot product of second-order tensors **A** and **B** is given by

$$\mathbf{A} : \mathbf{B} = a_{ij} b_{ij} \equiv \sum_i \sum_j a_{ij} b_{ij}.$$

In the previous examples, the number of indices in each of the components (e.g., one and two in the case of u_i and a_{ij}, respectively) indicates the order of the tensor.

As an alternative to the matrix notation, $\mathbf{y} = \mathbf{A}\mathbf{x}$, including Rules 2 and 3 allows the a system of linear equations to be written in an alternative compact form:

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_1 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots &= \qquad\qquad \vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}\right\} \Rightarrow y_i = a_{ij}x_j \equiv \sum_j^n a_{ij}x_j. \tag{3.1}$$

where $i = 1, \ldots, m$. Moreover, while the expression $y_i = a_{ij} x_j$ is valid, the expressions $M_{ij} u_j v_j + w_i$ and $T_{ijk} u_k + M_{ip}$ are invalid because the index j appears three times in $M_{ij} u_j v_j$, while the non-repeated index j in $T_{ijk} u_k$ doesn't match the non-repeated p of M_{ip}.

Rules 1 and 2, in particular, describe the separate treatment of *dummy* indicies – which appear no more than twice per term and are summed over – and *free* indices – which appear once per term and are not summed. In Eq. (3.1), i is a free index, while j is a dummy index.

The great advantage of using the summation convention is that all quantities in an expression become scalars able to be manipulated; (re-)ordered in whatever way that is convenient, although care must be taken when an expression involves operators.

3.2 The Kronecker-δ and Permutation Tensor ε

When writing vector expressions in the summation convention, two tensors appear so often that they are conventionally always given the same symbol: the Kronecker delta, δ and the permutation tensor, ε (*syn.*, Levi-Civita permutation symbol, anti-symmetric tensor).

The Kronecker delta is a rank 2 tensor defined by:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and is equivalent to the identify matrix, $\mathbf{I}$. In an expression, it has the effect of replacing one index with another, e.g.,

$$\begin{aligned} u_j &= u_i \delta_{ij}, \\ c_{ik} &= \delta_{ij} a_{jk}. \end{aligned}$$

The permutation symbol, ε, is a rank 3 tensor defined by:

$$\varepsilon_{ijk} = \begin{cases} +1, & ijk, \text{ even permutation: 123, 231, and 312} \\ -1, & ijk, \text{ odd permutation: 321, 213, 132} \\ 0, & \text{otherwise} \end{cases}$$

where, specifically, *even* refers to cyclic permutations of 123, *odd* refers to cyclic permutations of 321, and *otherwise* accounts for the twenty-one instances of repeated (equiv., absent) indices (e.g., 112 where 1 is repeated and 3 is absent).

Recall the vector cross product written as the determinant of a 3 × 3 matrix:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\hat{\imath} - (u_1 v_3 - u_3 v_1)\hat{\jmath} + (u_1 v_2 - u_2 v_1)\hat{k}.$$

This can also be determined using ε_{ijk} as follows $\mathbf{c} = \mathbf{u} \times \mathbf{v} \Rightarrow c_i = \varepsilon_{ijk} u_j v_k$. For illustration purposes, let $i = 1$ and consider every combination of j and k:

$$\begin{aligned} c_1 = \sum_k \sum_j \varepsilon_{1jk} u_j v_k =& \cancelto{0}{\varepsilon_{111} u_1 v_1} + \cancelto{0}{\varepsilon_{121} u_2 v_1} + \cancelto{0}{\varepsilon_{131} u_3 v_1} + \cdots \\ & \cancelto{0}{\varepsilon_{112} u_1 v_2} + \cancelto{0}{\varepsilon_{122} u_2 v_2} + \varepsilon_{132} u_3 v_2 + \cdots \\ & \cancelto{0}{\varepsilon_{113} u_1 v_3} + \varepsilon_{123} u_2 v_3 + \cancelto{0}{\varepsilon_{133} u_3 v_3} \\ =& u_2 v_3 - u_3 v_2. \end{aligned}$$

Similar results are achieved for $i = 1, 2$; however, the effort is greatly simplified in recalling that $\varepsilon_{ijk} = 0$ for repeated indices. Consider the cross product, $c_i = \varepsilon_{ijk} u_j v_k = c_1 \hat{\imath} + c_2 \hat{\jmath} + c_3 \hat{k}$, once again:

$$\begin{aligned} c_1 =& \varepsilon_{123} u_2 v_3 + \varepsilon_{132} u_3 v_2 = u_2 v_3 - u_3 v_2, \\ c_2 =& \varepsilon_{213} u_1 v_3 + \varepsilon_{231} u_3 v_1 = -(u_1 v_3 - u_3 v_1), \\ c_3 =& \varepsilon_{312} u_1 v_2 + \varepsilon_{321} u_2 v_1 = u_1 v_2 - u_2 v_1. \end{aligned}$$

Tensor notation allows for increased flexibility of the order in which factors are written than is permitted in vector notation. For example, $\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}$. In contrast $\varepsilon_{ijk} u_j v_k = \varepsilon_{ijk} v_k u_j = u_j v_k \varepsilon_{ijk}$ because the order of operation is dictated by the indices rather than the order in which the factors are written.

The permutation tensor and the Kronecker delta are related to each other through $\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$.

3.3 Summation Convention and Operators

Differentiation with respect to a space variable is written with the aid of a comma, e.g., $f_{,i} = \partial f / \partial x_i$. In the case of differentiation with respect to time, either the dot or comma notation may be used, e.g., $\partial^2 f / \partial t^2 = \ddot{f} = f_{,tt}$, where it is understood that no summation is implied by the double indices in $f_{,tt}$.

Vector operators are easily handled using summation convention. The vector operator ∇ can be thought of as a vector, e.g., in three dimensions $\nabla = [\partial_x \;\; \partial_y \;\; \partial_z]^{\mathrm{T}}$. The grad, div and curl operations in the summation convention look like:

$$\begin{aligned} \nabla \varphi &= \varphi_{,i} \\ \nabla^2 \varphi &= \varphi_{,ii} \\ \nabla \cdot \mathbf{v} &= v_{i,i} \\ \nabla \times \mathbf{v} &= \varepsilon_{ijk} v_{j,k}. \end{aligned}$$

Note that all operators in summation convention become scalar operators. The only thing to be careful of when dealing with operators in the summation convention is that unlike scalars, scalar operators cannot necessarily be freely re-ordered, since they must always appear to the left of whatever they operate on, e.g., $\nabla \times \mathbf{v} \neq \mathbf{v} \times \nabla$.